Conservation principles and action schemes in the synthesis of geometric concepts

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Diagrammatic reasoning

- Reasoning
- Learning
- Perception
- Design and creativity
- Theorem proving
- Ubiquitous in science and engineering

Diagrammatic reasoning

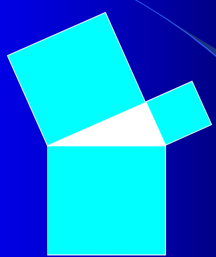
- How diagrammatic knowledge is represented
- What kind of inferences are supported by diagrams
- How external representations participate in this process

This is a problem in knowledge representation!

Some general questions about diagrams

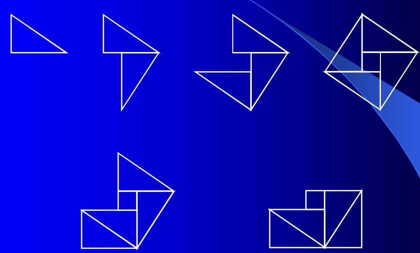
- What is their expressive power
- Why can they be interpreted so effectively
- What is the relation between logic and diagrammatic reasoning

Theorem of Pythagoras

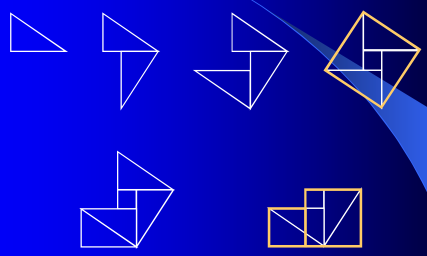


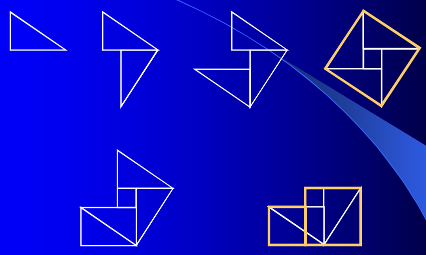
$$h^2 = a^2 + b^2$$

Bronowsky's proof



Bronowsky's proof





What are the mechanisms involved in this kind of reasoning?

A Challenge for AI

- Gelenter's GTP (late 50's): no account!
- Pineda (1989): The role of reinterpretations
- Barwise and Etchemendy: To illustrate heterogeneous reasoning (1990)
- Wang (1995): The need for generic descriptions
- Lindsay (1998): A demonstrator system
- Jamnik (1999): To illustrate a taxonomy of diagrammatic theorems

A Challenge for AI

- Pineda (2007):
  - A theory of diagrammatic reasoning
  - A semi-automatic proof of the theorem of Pythagoras
  - A semi-automatic proof of the theorem of the sum of the odds
  - A prototype program

The theory…

- Action schemes (a synthetic machinery)
- A notion of *re*-interpretation
- A geometric description machinery
- Conservation principles
- The arithmetic interpretation

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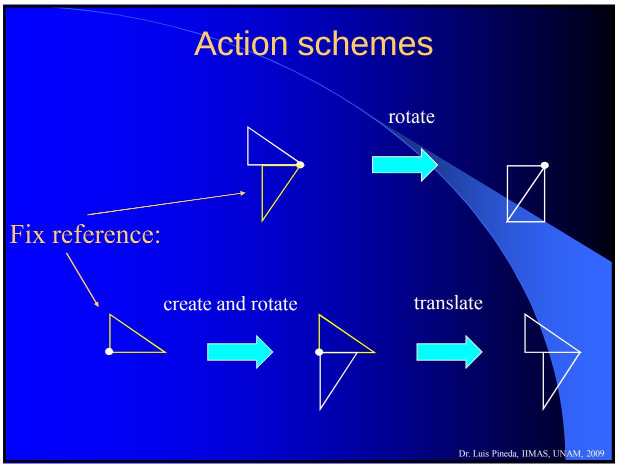
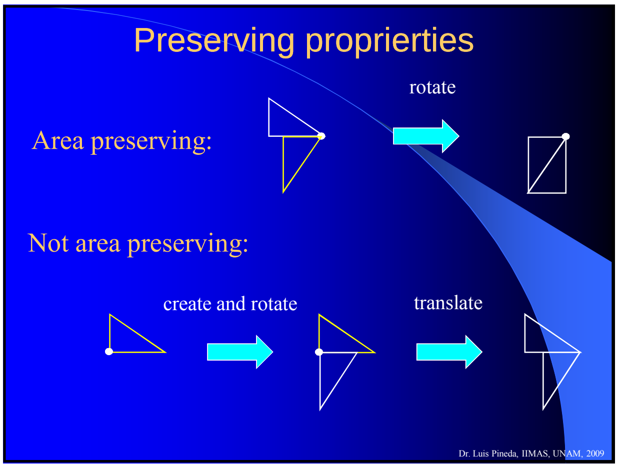
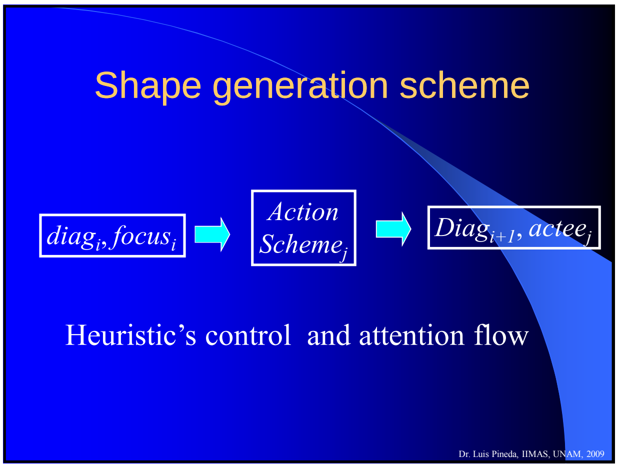

Action schemes
rotate
Fix reference:
create and rotate
translate
Dr. Luis Pineda, IIMAS, UNAM, 2009


Preserving proprierties
rotate
Area preserving:
Not area preserving:
create and rotate
translate
Dr. Luis Pineda, IIMAS, UNAM, 2009


Shape generation scheme
$diag_i, focus_i$
Action $Scheme_j$
$Diag_{i+1}, actee_j$
Heuristic's control and attention flow
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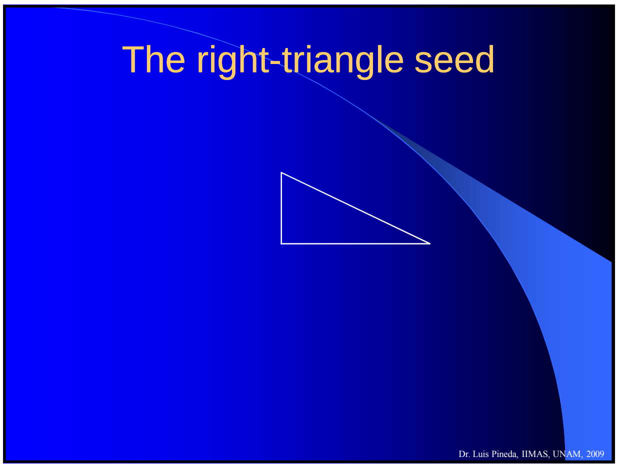
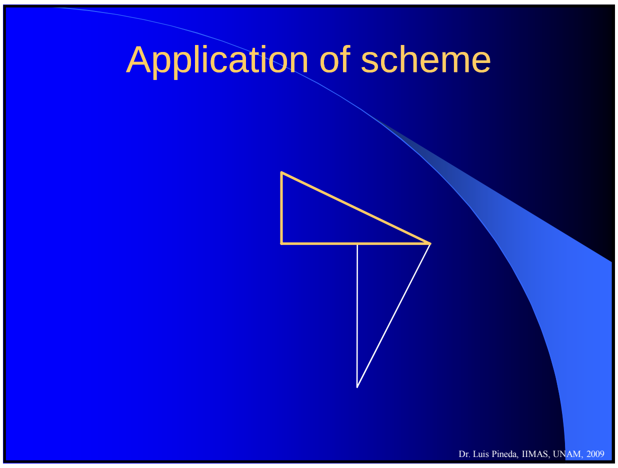

The right-triangle seed
Dr. Luis Pineda, IIMAS, UNAM, 2009


Application of scheme
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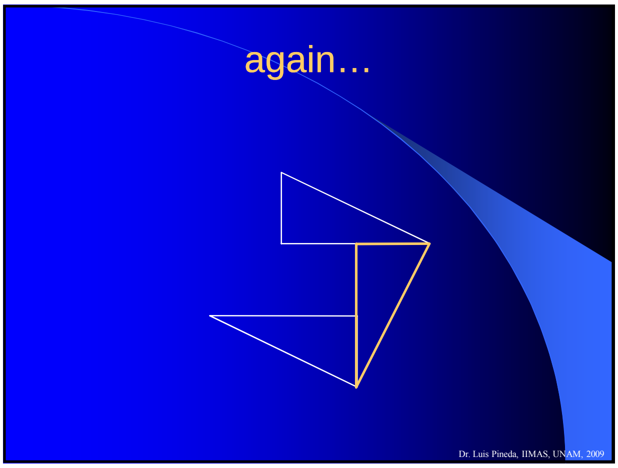
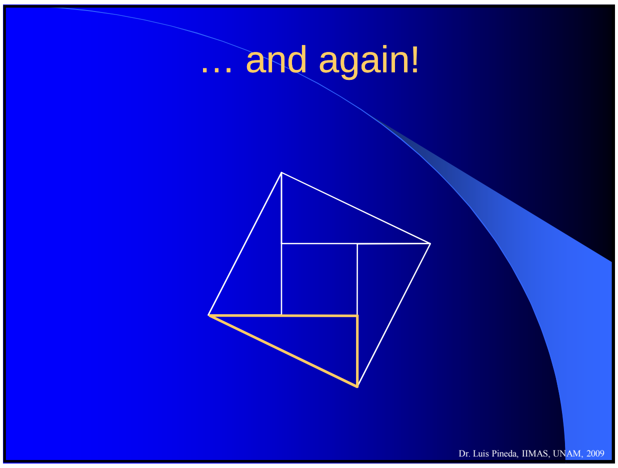
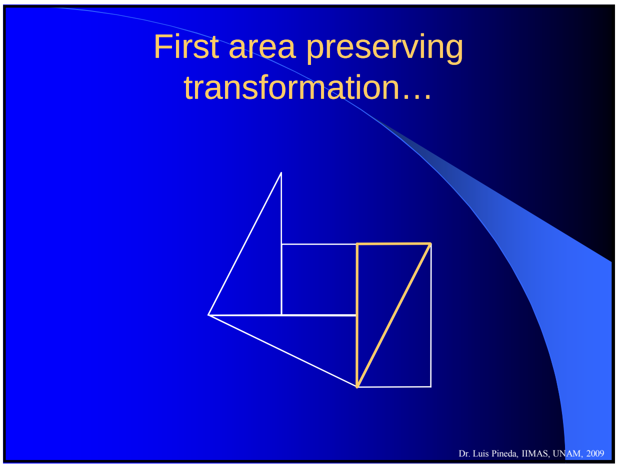
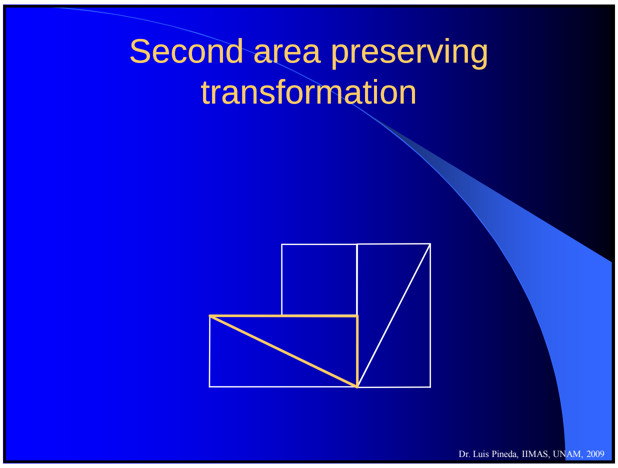
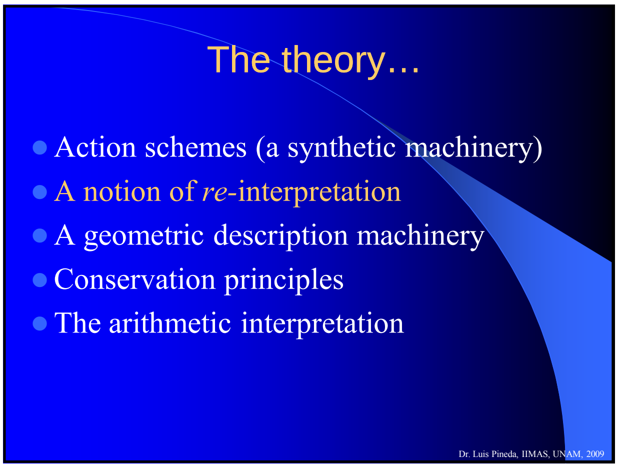

again…
Dr. Luis Pineda, IIMAS, UNAM, 2009


… and again!
Dr. Luis Pineda, IIMAS, UNAM, 2009


First area preserving transformation…
Dr. Luis Pineda, IIMAS, UNAM, 2009


Second area preserving transformation
Dr. Luis Pineda, IIMAS, UNAM, 2009


The theory…
Action schemes (a synthetic machinery)
A notion of *re*-interpretation
A geometric description machinery
Conservation principles
The arithmetic interpretation
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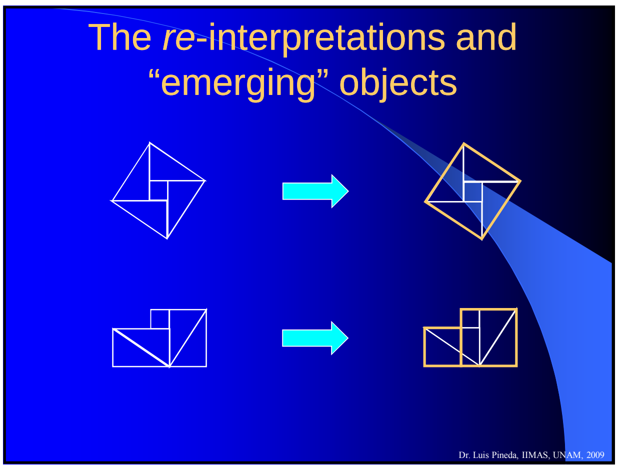
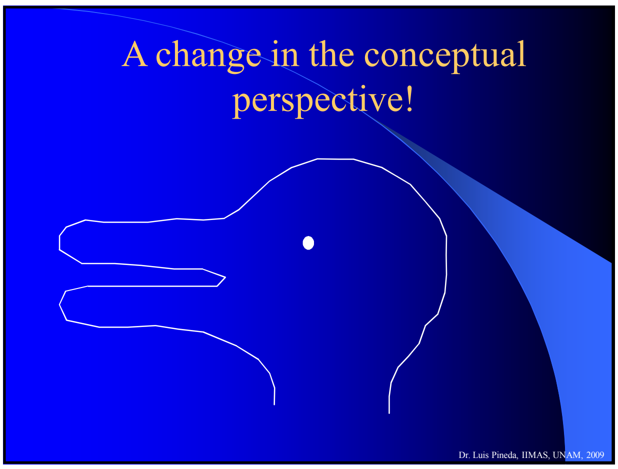

The *re*-interpretations and "emerging" objects
Dr. Luis Pineda, IIMAS, UNAM, 2009

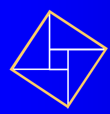

A change in the conceptual perspective!
Dr. Luis Pineda, IIMAS, UNAM, 2009

A problem of description…

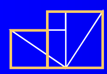
diagram → *Perceptual Scheme* → *Description*

A perceptual inference?

We need the relevant description



A square on the hypotenuse of a right-triangle



The union of a square on a right side of a right triangle and a square on the other right side of the same right triangle

The theory…

- Action schemes (a synthetic machinery)
- A notion of *re*-interpretation
- A geometric description machinery
- Conservation principles
- The arithmetic interpretation

Concepts

- Concepts (i.e. knowlegde objects) can be represented in computers
- Turing Machines campute functions
- So, concepts are represented through functions
- The challange is to find such functions
- In the present case, the functions representing geometric and arithmetic concepts that are expressed through diagrams!

Geometric description machinery

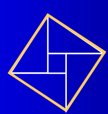
- A geometric signature to refer to geometric objects, properties and relations
- The functional abstractor operator to express geometric concepts
- A geometric descriptor operator to refer to (contextually dependent) emerging objects:

$$T <= f$$

  – If $f(A)$ is true $(T <= f) = T$ where T is a term of any geometric sort which contains (possible) variables in f

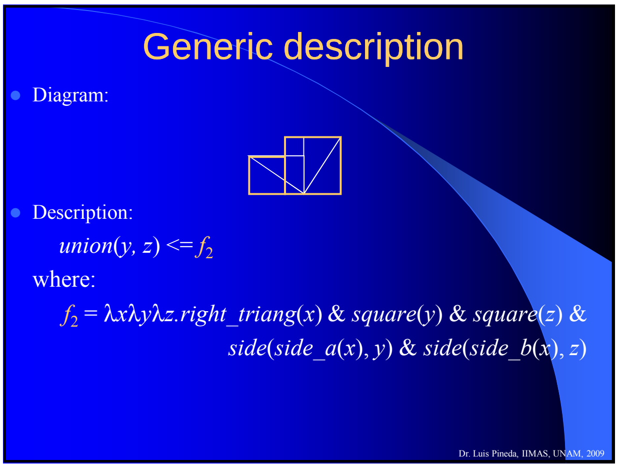
Generic description

- Diagram:



- Description: $y <= f_1$
- where:

$$f_1 = \lambda x \lambda y.right\_triang(x) \ \& \ \text{square}(y) \ \& \ side(hipotenuse(x), y)$$


Generic description
Diagram:
Description:
$union(y, z) <= f_2$
where:
$f_2 = \lambda x \lambda y \lambda z.right\_triang(x)$ & $square(y)$ & $square(z)$ &
$side(side\_a(x), y)$ & $side(side\_b(x), z)$
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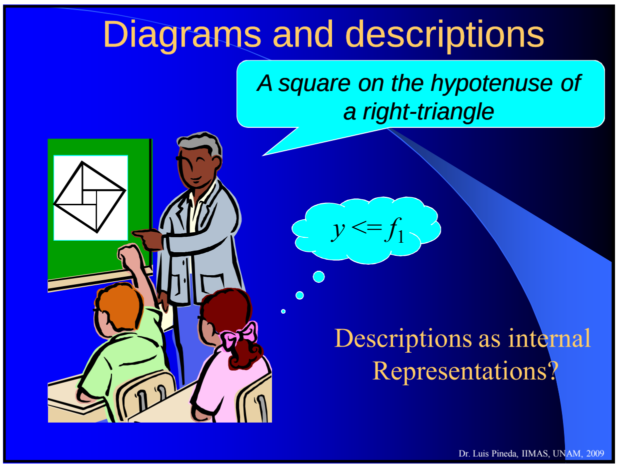
Diagrams and descriptions
A square on the hypotenuse of a right-triangle
$y <= f_1$
Descriptions as internal Representations?
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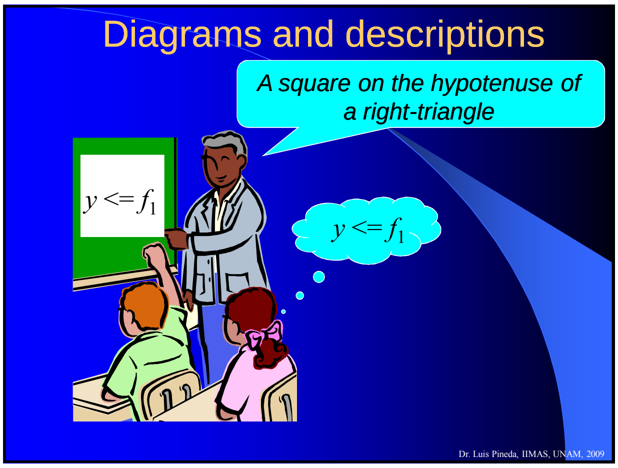
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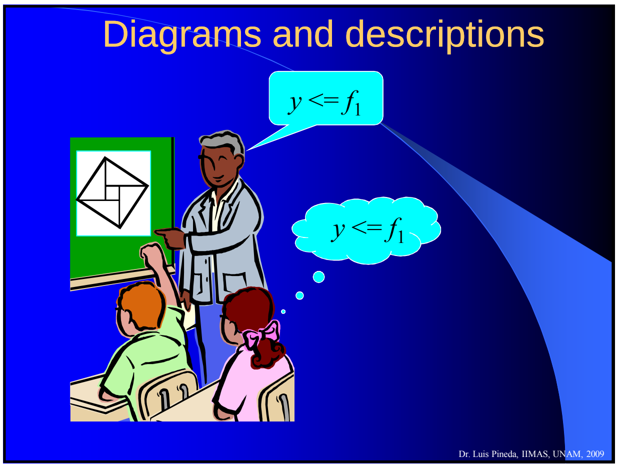
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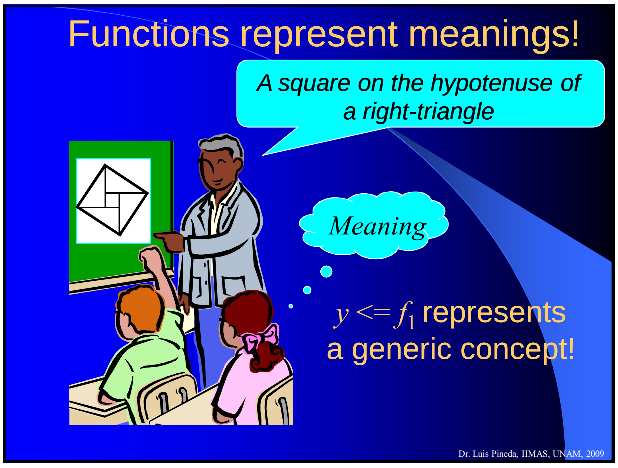
Functions represent meanings!
A square on the hypotenuse of a right-triangle
Meaning
$y <= f_1$ represents a generic concept!
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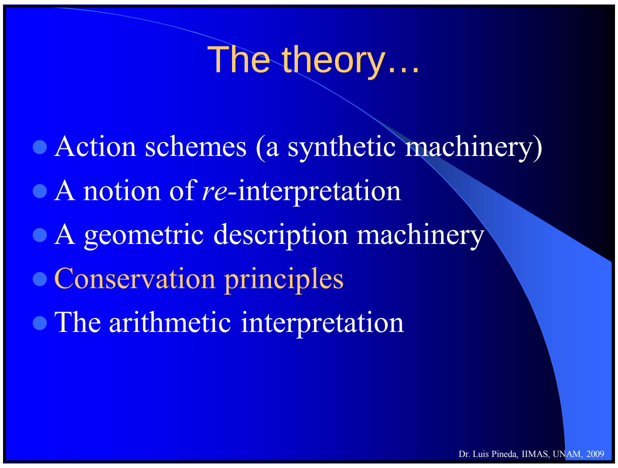
The theory…
Action schemes (a synthetic machinery)
A notion of *re*-interpretation
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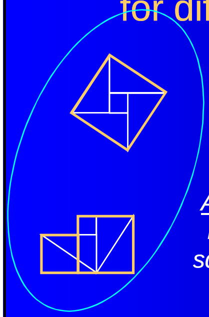
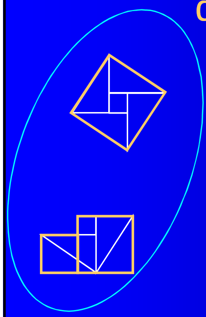
We need to state a property holds for different diagrams…



<u>Area of</u> a square on the hypotenuse of a right-triangle

<u>Area of</u> the union of a square on a right side of a right triangle and a square on the other right side of the same right triangle

This is a relation between generic descriptions…



$area(y <= f_1)$

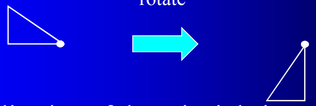
$area(union(y, z) <= f_2)$

Conservation principles

- Generalized concept of equality for geometrical properties
- <u>Global</u> principle of conservation of area:

$\lambda P \lambda Q(area(P) = area(Q))$

rotate



- The application of the principle is granted if the action scheme producing the transformation preserves the conservation property

Conservation principles

- Structured principle of conservation of area:

$\lambda P \lambda Q \lambda x(area(P(x)) = area(Q(x)))$

  - P and Q are generic descriptions of geometrical objects or configurations
  - x is a generic reference object
- An interpretative action (under the appropriate conditions) is represented by a functional application operation!

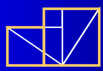
Synthesis of geometric concepts

$\lambda P \lambda Q \lambda x(area(P(x)) = area(Q(x))(y <= f_1)$

Synthesis of geometric concepts

$\lambda Q \lambda x(area(y <= f_1(x)) = area(Q(x))$

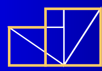
Synthesis of geometric concepts



$\lambda Q \lambda x(area(y <= f_1(x)) = area(Q(x))(union(y, z) <= f_2)$

The application is permitted if the the diagram is modified by an area preserving (sequence of) transformation

Synthesis of geometric concepts



$\lambda x(area(y <= f_1(x)) = area(union(y, z) <= f_2(x))$

The function representing the geometric concept of the Theorem of Pythagoras!

The geometric concept

$f_{TP} = \lambda x(area((w <= f_1)(x)) = area((union(y, z) <= f_2)(x))$

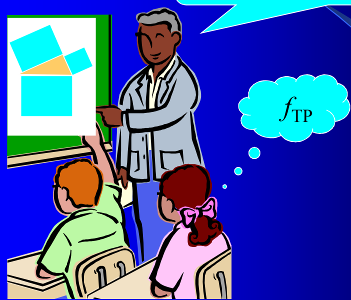
Where:

$f_1 = \lambda x \lambda y.right\_triang(x)$ & square(y) & $side(hipotenuse(x), y)$

$f_2 = \lambda x \lambda y \lambda z.right\_triang(x)$ & $square(y)$ & $square(z)$ & $side(side\_a(x), y)$ & $side(side\_b(x), z)$

The extension of the concept…

Are these in the Pythagorean relation?



The extension of the concept…

Are these?

Representation of meanings!

The area on the hypotenuse of a right triangle is the same as the area of the union of the squares on its right sides

The theory…

- Action schemes (a synthetic machinery)
- A notion of *re*-interpretation
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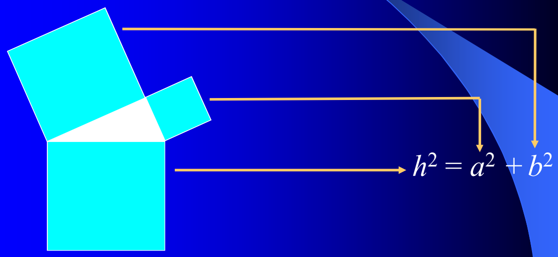
A mapping from the geometry into the arithmetic



The representation function

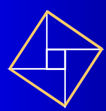
$\phi(x <= f) = \lambda u.u^2$ if the type of x *in* f is *sq*

$\phi(union) = +$

$\phi(g(y_1, y_2) <= f) = \phi(g)(\phi(y_1 <= f), \phi(y_2 <= f))$

The mapping

Diagram:

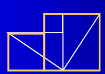


The arithmetic concept:

$\phi(y <= f_1) = \lambda u.u^2$

The mapping…

The diagram:



The arithmetic concept:

$\phi(union(y, z) <= f_2) = +(\lambda v.v^2, \lambda w.w^2)$

The mapping

- The geometric principle:
  - $\lambda P \lambda Q \lambda x(area(P(x)) = area(Q(x))$
- The arithmetic principle:
  - $\lambda P \lambda Q(P = Q)$
  - Concept of global aritmetic equality!

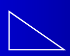
Diagrammatic Derivations

A three-tier tandem process

- The synthesis of geometric form
- The synthesis of the geometric concept
- The synthesis of the arithmetic concept

The seed...

- Diagram:



- Principle of conservation of area:
  - $\lambda P \lambda Q \lambda x(area(P(x)) = area(Q(x)))$
- Concept of the global arithmetic equality:
  - $\lambda P \lambda Q(P = Q)$

Synthesis of form

- Diagram:

- Principle of conservation of area:
  - $\lambda P \lambda Q \lambda x(area(P(x)) = area(Q(x)))$
- Concept of the global arithmetic equality:
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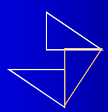
Synthesis of form

- Diagram:



- Principle of conservation of area:
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Synthesis of form

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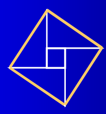
First reinterpretation

- Reinterpretations preserve area:



- Concepts construction:
  - $\lambda P \lambda Q \lambda x(area(P(x)) = area(Q(x))(w <= f_1)$
  - $\lambda P \lambda Q(P = Q)(\lambda u.u^2)$

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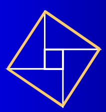
First reinterpretation

- Reinterpretation:



- Concepts construction:
  - $\lambda Q \lambda x(area(w <= f_1(x)) = area(Q(x))$
  - $\lambda Q(\lambda u.u^2 = Q)$

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Synthesis of form

- Diagram:

- Concepts construction:
  - $\lambda Q \lambda x(area(w <= f_1(x)) = area(Q(x))$
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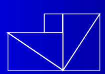
Synthesis of form

- Diagram:



- Concepts construction:
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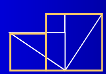
Second reinterpretation

- Reinterpretation:



- Concepts construction:
  - $\lambda Q \lambda x(area(w <= f_1(x)) = area(Q(x))(union(y, z) <= f_2)$
  - $\lambda Q(\lambda u.u^2 = Q)(+(\lambda v.v^2, \lambda w.w^2))$

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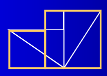
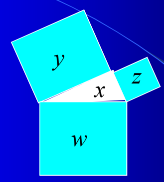
Second reinterpretation

- Reinterpretation:



- Concepts construction:
  - $\lambda x(area(w <= f_1(x)) = area(union(y, z) <= f_2(x))$
  - $\lambda u.u^2 = +(\lambda v.v^2, \lambda w.w^2)$

Program transformation rules



$$\lambda x.\lambda w.\lambda y, z.(area((w <= f_1)(x,w)) = area((union(y, z) <= f_2)(x,(y, z)))$$

$$\lambda u,v,w.u^2 = v^2 + w^2$$

Questions about diagrams

- What is their expressive power
- Why can they be interpreted so effectively
- What is the relation between logic and diagrammatic reasoning

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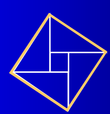
- Why can they be interpreted so effectively
- What is the relation between logic and diagrammatic reasoning

Diagrams and abstraction

- A common view is that diagrams are good for expressing concrete information but…
- There is a limitation in the abstractions that can be expressed
- The theory of graphical specificity (Stenning and Oberlander, 1995)

We can have concrete interpretations…

- Diagram:



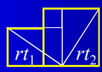
- Description:

  $sq_1 <= right\text{-}triangle(rt_1)$ & $square(sq_1)$ & $side(hipotenuse(rt_1), sq_1)$

… and deal with the ambiguity!

We can limit the expressive power of the representational language…

Diagram:



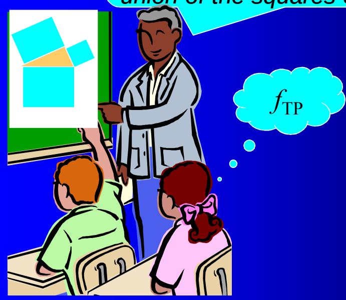
Descripción:

$union(sq_1, sq_2) <= right\_triang(rt_1)\ \&\ right\_triang(rt_2)\ \&$
$\mathrm{square}(sq_1)\ \&\ square(sq_2)\ \&$
$side(side\_a(rt_1), sq_1)\ \&$
$side(side\_b(rt_2), sq_2)$

and face the limitations of the medium!

Representation of meanings

The area on the hypotenuse of a right triangle is the same as the area of the union of the squares on its right sides



Through the *lambda calculus* we represent <u>interpretations of diagrams</u> …

NOT diagrams!!!

Diagrams and abstraction

- The present theory shows that diagrams can be given generic (fully abstract) interpretations!
- A representation is specified through:
  - The external symbols and configurations
  - <u>The interpretation process</u>
  - The language to represent the interpretations <u>does not need to have a limited expressivity</u> (e.g. propositional logic)
- Diagrammatic proofs are genuine proofs!

Questions about diagrams

- What is their expressive power
- Why can they be interpreted so effectively
- What is the relation between logic and diagrammatic reasoning

Reasoning with concrete representations

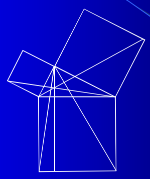
- Vision provides concrete interpretations of shapes directly
- Easy… if the problem has a concrete nature!
- Concrete problems can often be expressed through diagrams
- But, if the problem demands abstraction (e.g. an infinite number of instances) concrete resources (memory and computational time) run out very quickly!

Abstractions capture change implicitly!

- Two dimensions of change:
  - The parameters of the diagrammatic objects
  - Different diagrammatic configurations that have the same description (i.e. equivalent in relation to the task)

Abstractions account for equivalent objects!

- Diagram:

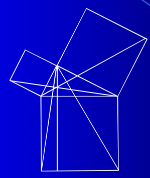


- Description: $y \Leftarrow f_1$
- where:

  $f_1 = \lambda x \lambda y.right\_triang(x)\ \&\ \text{square}(y)\ \&\ side(hipotenuse(x), y)$

Abstractions account for equivalent objects!

- Diagram:



- Description:

  $union(y, z) \Leftarrow f_2$

  where:

  $f_2 = \lambda x \lambda y \lambda z.right\_triang(x)\ \&\ square(y)\ \&\ square(z)\ \&\ side(side\_a(x), y)\ \&\ side(side\_b(x), z)$

Diagrammatic reasoning is monotonic!

- In spite of the change in the geometric form and regardless the values of the parameters of diagrammatic objects, the synthesis of the geometric and arithmetic processes is monotonic

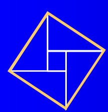
Reading a diagrammatic sequence!

- Incremental interpretation:
  - *every man is mortal*
  - $\lambda P \lambda Q \lambda x(P(x) \supset Q(x))(\text{man})(\text{mortal})$
  - $\lambda Q \lambda x(\text{man}(x) \supset Q(x))(\text{mortal})$
  - $\lambda x(\text{man}(x) \supset \text{mortal}(x))$
- There is not a change to account for!
- Natural language quantifiers can be seen as conservation principles!

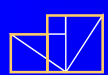
Reasoning with abstractions is easy!

- Abstractions are small finite representational objects (that represent interpretations) that can be used in thought process as units, but have a very large, perhaps infinite, extension

What is hard is to produce the relevant abstractions!

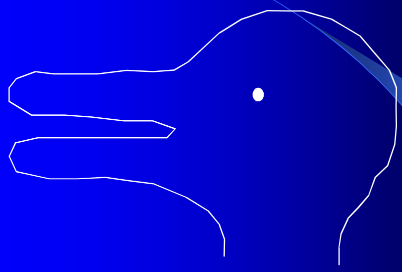


A square on the hypotenuse of a right-triangle



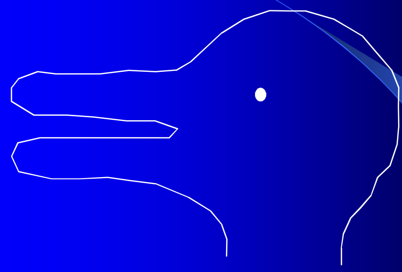
The union of a square on a right side of a right triangle and a square on the other right side of the same right triangle

Abstracting on concrete descriptions?



$duck_1 <= duck(duck_1)$?

Constructing the abstraction directly!



$x <= \lambda x.duck(x)$

Generation of abstract descriptions …

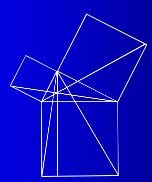
- The extensional representation
- Visualisations (i.e. Reinterpretations)
- Domain knowlege (e.g. Geometry)
- Knowledge about the aims of the task (e.g. theorem proving and discovery)

Perceptual inference

Questions about diagrams

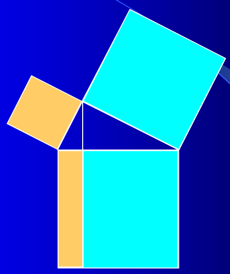
- What is their expressive power
- Why can they be interpreted so effectively
- What is the relation between logic and diagrammatic reasoning

The axiomatic method



Proposition 47, Euclid's Elements

The axiomatic method



A simpler problem!

Dr. Luis Pineda, IIMAS, UNAM, 2009

Reinterpretations

- Enrich the problem-solving space
- Interesting emerging objects belong to the enriched space
- The recognition of emerging objects depends on the interpretation process, but also on the nature of the external representation!
- The process is genuinely synthetic and synthesized objects cannot be found through analysis!

Dr. Luis Pineda, IIMAS, UNAM, 2009

The paper:

Luis A. Pineda, Conservation principles and action schemes in the synthesis of geometric concepts, *Artificial Intelligence* 171 (March, 2007) 197-238.

Dr. Luis Pineda, IIMAS, UNAM, 2009

Thanks very much!

Dr. Luis Pineda, IIMAS, UNAM, 2009