

Session 11

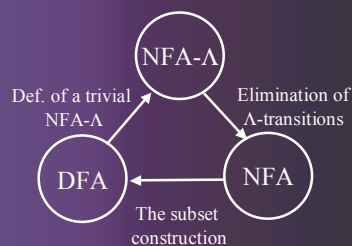
Kleene's Theorem

Kleene's Theorem

- A language L over the alphabet Σ is regular (can be expressed through a regular expression) if and only if there is an FA with input alphabet Σ that accepts L
 - Part 1 (only if): If there is a RE for L then there is a FA accepting L
 - Part 2 (if): If there is a FA accepting L then there is a RE for L

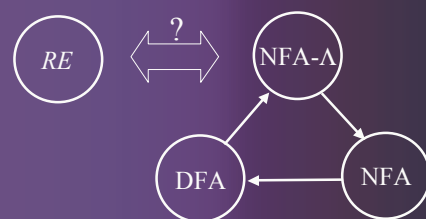
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The Family of FA



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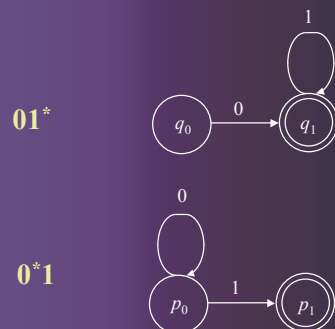
RE and FA



We can use any of these as the three are equivalent, but a particular choice may be better than the others!

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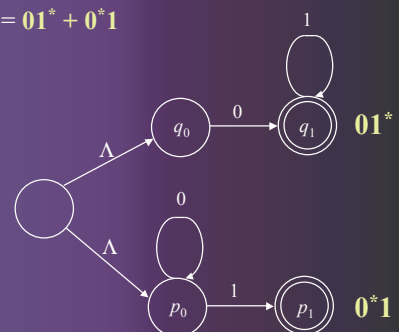
Similarity of RE and NFA-λ



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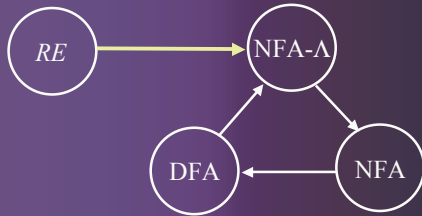
Similarity of RE and NFA-λ

- $L_1 \cup L_2 = 01^* + 0^*1$



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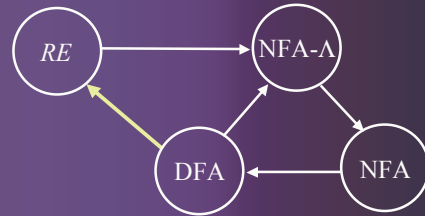
The simplest strategy



If there are FA for the parts of a RE , we can compose a composite FA corresponding to the RE out of the simpler FA's (with the help of Λ -transitions)

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The simplest strategy



In this direction, we need to construct a RE that accounts for all strings of any length that are accepted by the corresponding FA

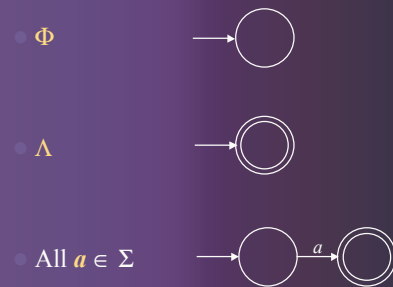
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Proof of the theorem: Part 1

- All RE are built out of
 - Φ , Λ and All $a \in \Sigma$
- And the composition operators
 - $E+F$, EF and E^*
- Automata Composition:
 - Define a FA for the basic parts
 - Define the form of FA for the three operators
 - Construct the FA in tandem with the corresponding RE

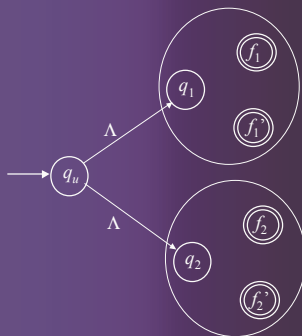
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Proof of the theorem: Part 1



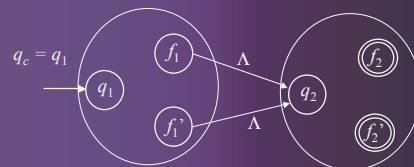
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The union pattern: $E + F$



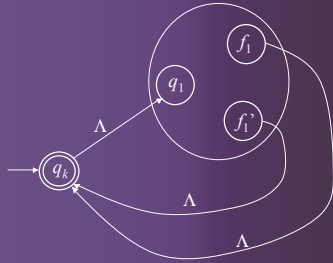
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The concatenation pattern: EF



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The closure pattern: E^*



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The construction

- With the FA for the basic RE
- With FA schemes for the composite RE
- Construct the composite FA in tandem with the structure of the RE !
- An example:

Construct the FA for the $RE: (01)^* + (10)^*$

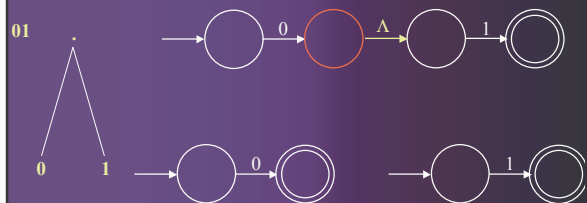
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Example: the FA for $(01)^*$



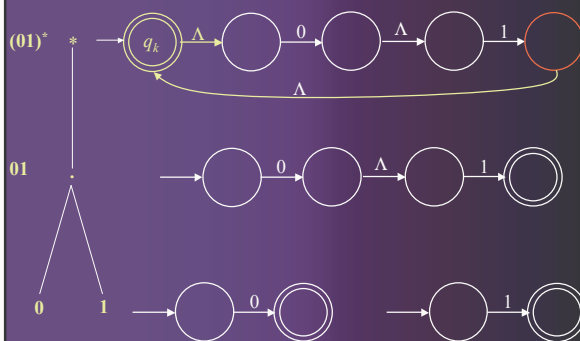
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Example: the FA for $(01)^*$



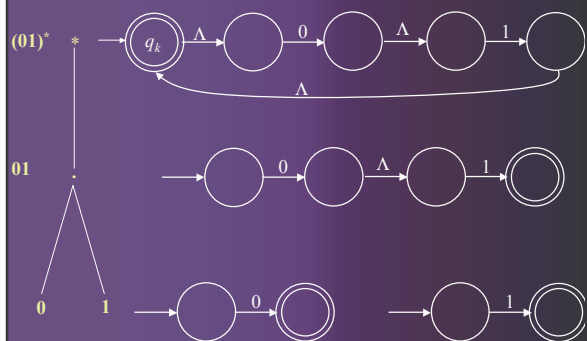
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Example: the FA for $(01)^*$

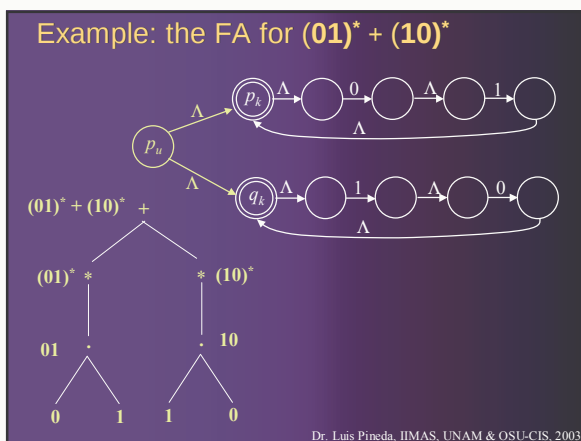
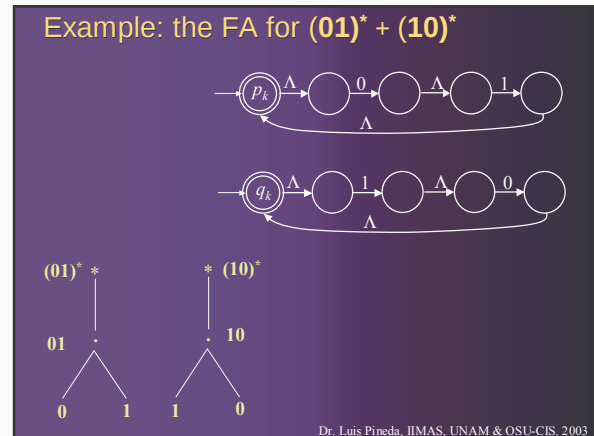
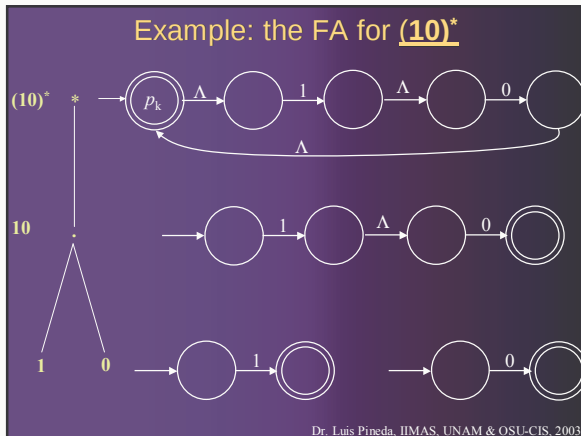


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Example: the FA for $(01)^*$



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Kleene's Theorem

✓Part 1 (only if): If there is a regular language L (expressed by RE) then there is a FA accepting L

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Kleene's Theorem

✓Part 1 (only if): If there is a regular language L (expressed by RE) then there is a FA accepting L

●Part 2 (if): If there is a FA accepting L then there is a regular expression for L

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The sets accepted by a FA

- Let $L \subseteq \Sigma^*$ be accepted by the FA $M = (Q, \Sigma, q_0, A, \delta)$.
 - this is:

$$L = \{x \in \Sigma^* \mid \delta^*(q_0, x) \in A\}$$
 - If $A = \{q_i, \dots, q_j\}$ then L is the union of a finite number of sets of the form (one for each accepting state):

$$L = \{x \in \Sigma^* \mid \delta^*(q_0, x) = q\}$$
 - Consequently there must be a RE denoting each of these sets!
 - Also, if p and q are states of a FA, there must be a RE for the set:

$$L(p, q) = \{x \in \Sigma^* \mid \delta^*(p, x) = q\}$$

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The sets accepted by a FA

- The *RE* corresponding to a FA will be:
 - The union of the *RE*

$$L(q_0, q) = \{x \in \Sigma^* \mid \delta^*(q_0, x) = q\}$$
for every $q \in A = \{q_1, \dots, q_f\}$
- In order to find the *RE* corresponding to this set, we construct the *RE* for the languages recognized by all possible paths of the form:

$$L(p, q) = \{x \in \Sigma^* \mid \delta^*(p, x) = q\}$$

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Path going through a state

- Let's assign a natural number n to every state of a FA
- For a string $x \in \Sigma^*$, x represents a path from state p to state q through state s if there are non null strings y and z so that
 - $x = yz$ $\delta^*(p, y) = s$ $\delta^*(s, z) = q$
- A path can go from or to a state without going through it

$$p \xrightarrow{y} s \xrightarrow{z} q$$
- Let the language of all paths that go through a state whose number is not higher than j be:

$$L(p, q, j) \text{ where } j \geq 0$$
- There are n states, so $L(p, q, n) = L(p, q)$

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Proof that a language is regular

- The induction:
 - We have to show that $L(p, q, n)$ is regular, and this will be the case if $L(p, q, j)$ is regular for every j such that $0 \leq j \leq n$
- The basis:
 - We need to show that $L(p, q, 0)$ is regular
 - This is, the languages of paths that go through no state in a FA are regular:

$$L(p, q, 0) \subseteq \Sigma \cup \{\Lambda\}$$
 - $L(p, q, 0)$ is regular because the number of states is finite, so the number of paths!

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Proof that a language is regular

- The induction hypothesis:
 - $0 \leq k$
 - for every p and q satisfying $0 \leq p$ and $q \leq n$ the language $L(p, q, k)$ is regular
 - we want to show that $L(p, q, k+1)$ is regular
- A string x is in $L(p, q, k+1)$ if it represents a path from p to q that goes through a state no higher than $k+1$; there are two ways this can happen:
 - Case 1:** The strings does not go through state $k+1$, so it does not go through a state higher than k , then

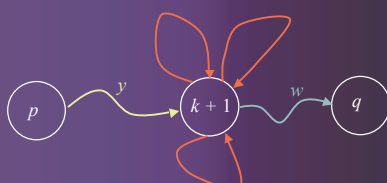
$$x \in L(p, q, k)$$

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Proof that a language is regular

- Case 2:** The strings goes through state $k+1$, but not through a state whose number is higher than $k+1$; for this:

$$x = yzw$$

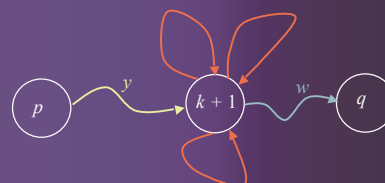


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Proof that a language is regular

- $x = yzw$
 - $y \in L(p, k+1, k)$ Reaches state $k+1$
 - $z \in L(k+1, k+1, k)^*$ Goes through state $k+1$ any times
 - $w \in L(k+1, q, k)$ Starts in state $k+1$
- x is constructed with concatenation and *-closure:

$$x \in L(p, k+1, k)L(k+1, k+1, k)^*L(k+1, q, k)$$



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The two cases together!

- Case 1:
– $x \in L(p, q, k)$
- Case 2:
– $x \in L(p, k+1, k)L(k+1, k+1, k)^*L(k+1, q, k)$
- Then
 $L(p, q, k+1) =$
 $L(p, q, k) \cup L(p, k+1, k)L(k+1, k+1, k)^*L(k+1, q, k)$
- $L(p, q, k+1)$ is constructed with:
 - union
 - concatenation
 - *-closure!
- Then, $L(p, q, k+1)$ is regular!

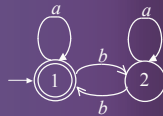
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So...?

- The proof of the theorem provides an algorithm to find the RE corresponding to any FA:
- $L(p, q, 0) = \begin{cases} \{a \in \Sigma \mid \delta(p, a) = q\} & \text{if } p \neq q \\ \{a \in \Sigma \mid \delta(p, a) = q\} \cup \{\Lambda\} & \text{if } p = q \end{cases}$
- $L(p, q, k+1) =$
 $L(p, q, k) \cup L(p, k+1, k)L(k+1, k+1, k)^*L(k+1, q, k)$
- $L(p, q, n) = L(p, q)$ and $L = \bigcup_{q \in A} L(q_0, q)$

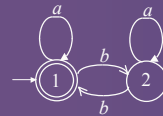
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What is RE corresponding to a FA?



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The RE corresponding to a FA

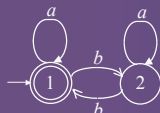


$(a^*ba^*ba^*)^* ?$
 $a^* + (ba^*ba^*)^* ?$
 $a^* + (a^*ba^*b)^* ?$

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The basic case: $r(p, q, 0)$

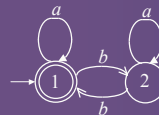
$$L(p, q, 0) = \begin{cases} \{a \in \Sigma \mid \delta(p, a) = q\} & \text{if } p \neq q \\ \{a \in \Sigma \mid \delta(p, a) = q\} \cup \{\Lambda\} & \text{if } p = q \end{cases}$$



p	$r(p, 1, 0)$	$r(p, 2, 0)$
1	$a + \Lambda$	b
2	b	$a + \Lambda$

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The inductive part: $r(p, q, k+1)$



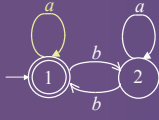
p	$r(p, 1, k+1)$	$r(p, 2, k+1)$
1		
2		

$$L(p, q, k+1) =$$

$$L(p, q, k) \cup L(p, k+1, k)L(k+1, k+1, k)^*L(k+1, q, k)$$

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The inductive part: $r(p,q,1)$



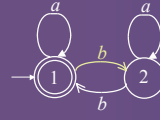
p	$r(p,1,1)$	$r(p,2,1)$
1	a^*	
2		

$$L(p, q, k+1) = L(p, q, k) \cup L(p, k+1, k)L(k+1, k+1, k)^*L(k+1, q, k)$$

$$\begin{aligned} k=0, k+1=1 \\ L(1, 1, 1) &= L(1, 1, 0) \cup L(1, 1, 0)L(1, 1, 0)^*L(1, 1, 0) \\ &= (a + \Lambda) + (a + \Lambda)(a + \Lambda)^*(a + \Lambda) \\ &= a^* \end{aligned}$$

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The inductive part: $r(p,q,1)$



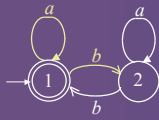
p	$r(p,1,1)$	$r(p,2,1)$
1	a^*	$b + a^*b$
2		

$$L(p, q, k+1) = L(p, q, k) \cup L(p, k+1, k)L(k+1, k+1, k)^*L(k+1, q, k)$$

$$\begin{aligned} k=0, k+1=1 \\ L(1, 2, 1) &= L(1, 2, 0) \cup L(1, 1, 0)L(1, 1, 0)^*L(1, 2, 0) \\ &= b + (a + \Lambda)(a + \Lambda)^*b \\ &= b + a^*b \end{aligned}$$

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The inductive part: $r(p,q,1)$



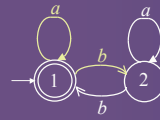
p	$r(p,1,1)$	$r(p,2,1)$
1	a^*	$b + a^*b$
2		

$$L(p, q, k+1) = L(p, q, k) \cup L(p, k+1, k)L(k+1, k+1, k)^*L(k+1, q, k)$$

$$\begin{aligned} k=0, k+1=1 \\ L(1, 2, 1) &= L(1, 2, 0) \cup L(1, 1, 0)L(1, 1, 0)^*L(1, 2, 0) \\ &= b + (a + \Lambda)(a + \Lambda)^*b \\ &= b + a^*b \end{aligned}$$

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Simplifying...



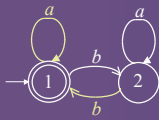
p	$r(p,1,1)$	$r(p,2,1)$
1	a^*	a^*b
2		

$$L(p, q, k+1) = L(p, q, k) \cup L(p, k+1, k)L(k+1, k+1, k)^*L(k+1, q, k)$$

$$\begin{aligned} k=0, k+1=1 \\ L(1, 2, 1) &= L(1, 2, 0) \cup L(1, 1, 0)L(1, 1, 0)^*L(1, 2, 0) \\ &= b + (a + \Lambda)(a + \Lambda)^*b \\ &= b + a^*b = a^*b \end{aligned}$$

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The inductive part: $r(p,q,1)$



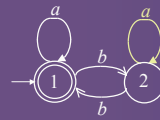
p	$r(p,1,1)$	$r(p,2,1)$
1	a^*	a^*b
2	ba^*	

$$L(p, q, k+1) = L(p, q, k) \cup L(p, k+1, k)L(k+1, k+1, k)^*L(k+1, q, k)$$

$$\begin{aligned} k=0, k+1=1 \\ L(2, 1, 1) &= L(2, 1, 0) \cup L(2, 1, 0)L(1, 1, 0)^*L(1, 1, 0) \\ &= b + b(a + \Lambda)^*(a + \Lambda)^* \\ &= b + ba^* = ba^* \end{aligned}$$

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The inductive part: $r(p,q,1)$



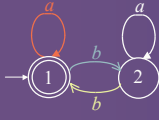
p	$r(p,1,1)$	$r(p,2,1)$
1	a^*	a^*b
2	ba^*	$\Lambda + a + ba^*b$

$$L(p, q, k+1) = L(p, q, k) \cup L(p, k+1, k)L(k+1, k+1, k)^*L(k+1, q, k)$$

$$\begin{aligned} k=0, k+1=1 \\ L(2, 2, 1) &= L(2, 2, 0) \cup L(2, 1, 0)L(1, 1, 0)^*L(1, 2, 0) \\ &= (a + \Lambda) + b(a + \Lambda)^*b \\ &= a + \Lambda + ba^*b \end{aligned}$$

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The inductive part: $r(p,q,1)$



p	$r(p,1,1)$	$r(p,2,1)$
1	a^*	a^*b
2	ba^*	$\Lambda + a + ba^*b$

$$L(p, q, k+1) = L(p, q, k) \cup L(p, k+1, k)L(k+1, k+1, k)^*L(k+1, q, k)$$

$$\begin{aligned} k &= 0, k+1 = 1 \\ L(2, 2, 1) &= L(2, 2, 0) \cup L(2, 1, 0)L(1, 1, 0)^*L(1, 2, 0) \\ &= (a + \Lambda) + b(a + \Lambda)^*b \\ &= a + \Lambda + ba^*b \end{aligned}$$

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The inductive part: $r(p,q,2)$

p	$r(p,1,1)$	$r(p,2,1)$	p	$r(p,1,2)$	$r(p,2,2)$
1	a^*	a^*b	1	RE_{11-2}	
2	ba^*	$\Lambda + a + ba^*b$	2		

$$L(p, q, k+1) = L(p, q, k) \cup L(p, k+1, k)L(k+1, k+1, k)^*L(k+1, q, k)$$

$$\begin{aligned} k &= 1, k+1 = 2 \\ L(1, 1, 2) &= L(1, 1, 1) \cup L(1, 2, 1)L(2, 2, 1)^*L(2, 1, 1) \\ &= a^* + (a^*b)(\Lambda + a + ba^*b)^*(ba^*) \\ &= RE_{11-2} \end{aligned}$$

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The inductive part: $r(p,q,2)$

p	$r(p,1,1)$	$r(p,2,1)$	p	$r(p,1,2)$	$r(p,2,2)$
1	a^*	a^*b	1	RE_{11-2}	RE_{12-2}
2	ba^*	$\Lambda + a + ba^*b$	2		

$$L(p, q, k+1) = L(p, q, k) \cup L(p, k+1, k)L(k+1, k+1, k)^*L(k+1, q, k)$$

$$\begin{aligned} k &= 1, k+1 = 2 \\ L(1, 2, 2) &= L(1, 2, 1) \cup L(1, 2, 1)L(2, 2, 1)^*L(2, 1, 1) \\ &= a^*b + (a^*b)(\Lambda + a + ba^*b)^*(a + ba^*b) \\ &= RE_{12-2} \end{aligned}$$

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The inductive part: $r(p,q,2)$

p	$r(p,1,1)$	$r(p,2,1)$	p	$r(p,1,2)$	$r(p,2,2)$
1	a^*	a^*b	1	RE_{11-2}	RE_{12-2}
2	ba^*	$\Lambda + a + ba^*b$	2	RE_{21-2}	

$$L(p, q, k+1) = L(p, q, k) \cup L(p, k+1, k)L(k+1, k+1, k)^*L(k+1, q, k)$$

$$\begin{aligned} k &= 1, k+1 = 2 \\ L(2, 1, 2) &= L(2, 1, 1) \cup L(2, 2, 1)L(2, 2, 1)^*L(2, 1, 1) \\ &= ba^* + (a + ba^*b)(\Lambda + a + ba^*b)^*(ba^*) \\ &= RE_{21-2} \end{aligned}$$

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The inductive part: $r(p,q,2)$

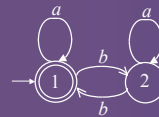
p	$r(p,1,1)$	$r(p,2,1)$	p	$r(p,1,2)$	$r(p,2,2)$
1	a^*	a^*b	1	RE_{11-2}	RE_{12-2}
2	ba^*	$\Lambda + a + ba^*b$	2	RE_{21-2}	RE_{22-2}

$$L(p, q, k+1) = L(p, q, k) \cup L(p, k+1, k)L(k+1, k+1, k)^*L(k+1, q, k)$$

$$\begin{aligned} k &= 1, k+1 = 2 \\ L(2, 2, 2) &= L(2, 2, 1) \cup L(2, 2, 1)L(2, 2, 1)^*L(2, 2, 1) \\ &= (a + ba^*b) + (a + ba^*b)(\Lambda + a + ba^*b)^*(a + ba^*b) \\ &= RE_{22-2} \end{aligned}$$

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The inductive part: $r(p,q)$



p	$r(p,1,2)$	$r(p,2,2)$
1	RE_{11-2}	RE_{12-2}
2	RE_{21-2}	RE_{22-2}

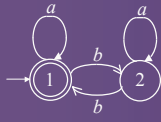
The end of the induction: $(p, q, n) = L(p, q)$ and

$$L = \bigcup_{q \in A} L(q_0, q)$$

$$\begin{aligned} L(1, 1, 2) &= L(1, 1, 1) \cup L(1, 2, 1)L(2, 2, 1)^*L(2, 1, 1) \\ &= a^* + (a^*b)(\Lambda + a + ba^*b)^*(ba^*) \end{aligned}$$

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The inductive part: $r(p,q)$



$$(a^*ba^*ba^*)^*?$$

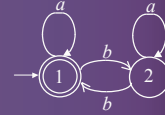
$$a^* + (ba^*ba^*)^*?$$

$$a^* + (a^*ba^*b)^*?$$

$$L(1, 1, 2) = a^* + (a^*b)(\Lambda + a + ba^*b)^*(ba^*)$$

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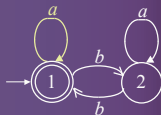
The inductive part: $r(p,q)$



$$L(1, 1, 2) = a^* + (a^*b)(\Lambda + a + ba^*b)^*(ba^*)$$

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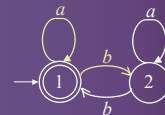
The inductive part: $r(p,q)$



$$L(1, 1, 2) = a^* + (a^*b)(\Lambda + a + ba^*b)^*(ba^*)$$

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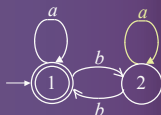
The inductive part: $r(p,q)$



$$L(1, 1, 2) = a^* + (a^*b)(\Lambda + a + ba^*b)^*(ba^*)$$

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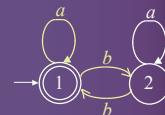
The inductive part: $r(p,q)$



$$L(1, 1, 2) = a^* + (a^*b)(\Lambda + a + ba^*b)^*(ba^*)$$

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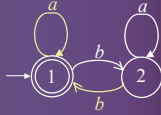
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$$L(1, 1, 2) = a^* + (a^*b)(\Lambda + a + ba^*b)^*(ba^*)$$

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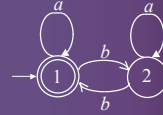
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$$L(1, 1, 2) = a^* + (a^*b)(\Lambda + a + ba^*b)^*(ba^*)$$

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The inductive part: $r(p,q)$



$$L(1, 1, 2) = a^* + (a^*b)(\Lambda + a + ba^*b)^*(ba^*)$$

The expression that reflects better the underlying structure of the FA

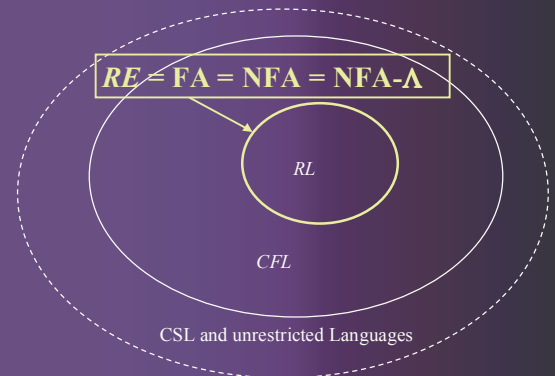
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Kleene's Theorem

- ✓ Part 1 (only if): If there is a regular language L (expressed by RE) then there is a FA accepting L
- ✓ Part 2 (if): If there is a FA accepting L then there is a regular expression for L

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Kleene's Theorem



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