Session 12

Minimal Automata

What property makes a language regular?

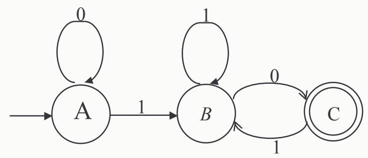
- Kleen's theorem: if we have a RE describing L or a FA accepting L, then L is regular
- But, what if we have a language described by some other means:
  - $L = \{0^n1^n \in \Sigma^* \mid n > 0\}$
  - Is this language regular?
- We know that if there are n classes of strings in L, a FA for recognizing L needs to have at least n states
  - All FA have a finite set of states
  - If the number of classes of strings of L is finite, there is a FA for recognizing L, and also a *RE* for describing L

Using the classes of strings

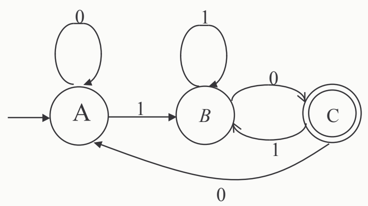
- We can tell what is the FA from the set of classes!
- A machine for the language of strings ending with 10:
  - Class A: The string needs 10 to be in the language
  - Class B: The string needs a 0 to be in the language
  - Class C: The string is in the language!

From the finite set of classes to the FA…

- A machine for strings ending with 10:
  - Class A: The string needs 10 to be in the language
  - Class B: The string needs a 0 to be in the language
  - Class C: The string is in the language!



Very sorry… it can get tricky!



- Fortunately, there is an algorithm for producing a FA from a finite set of classes of strings!

The classes of strings!

- We study the strings in the language directly: we don't have a FA or a *RE*!
- First, we need to identify the classes of strings:
  - Class A: The string needs 10 to be in the language: Λ, 0 or ends in 00
  - Class B: The string needs a 0 to be in the language: 1, 11 or ends in 01
  - Class C: The string ends with 10 and it is in the language!
- More generally, there is a distinguishing string z to get strings of each class into the language:

$$L/x = \{z \in \Sigma^* \mid xz \in L\}$$

Distinguishing strings

- Let L be a language in Σ^*, and $x \in \Sigma^*$
- Let L/x a set of strings such that:
  - $L/x = \{z \in \Sigma^* \mid xz \in L\}$
- Two strings x and y are distinguishable with respect to L if
  - $L/x \neq L/y$
- Any string z such that $xz \in L$ and $yz \notin L$ or vice versa, is said to distinguish x and y with respect to L
- If $L/x = L/y$, x and y are indistinguishable (or equivalent) with respect to L

The classes of strings!

- Class A:
  - Λ
  - 0
  - Σ^*00
- Class B:
  - 1
  - 11
  - Σ^*01
- Class C:
  - Σ^*10

Distinguishing the classes of strings!

- $z = 10$ and $L/x = \{z \in \Sigma^* \mid xz \in L\}$
  - $\Lambda 10$
  - 010
  - Σ^*0010
- $z = 0$ and $L/y = \{z \in \Sigma^* \mid yz \in L\}$
  - 10
  - 110
  - Σ^*010
- $z = \Lambda$ and $L/u = \{z \in \Sigma^* \mid uz \in L\}$
  - $\Sigma^*10\Lambda$
- We have one z for each class of strings!

distinguishing *L/x* from *L/y*

- Let $z = 0$
  - $L/x = \{z \in \Sigma^* \mid xz \notin L\}$
    - $\Lambda 0$
    - 00
    - Σ^*000
  - $L/y = \{z \in \Sigma^* \mid yz \in L\}$
    - 10
    - 110
    - Σ^*010
- $L/x \neq L/y$

distinguishing *L/x* from *L/u*

- Let $z = \Lambda$
  - $L/x = \{z \in \Sigma^* \mid xz \notin L\}$
    - $\Lambda\Lambda$
    - 0Λ
    - $\Sigma^*00\Lambda$
  - $L/u = \{z \in \Sigma^* \mid uz \in L\}$
    - $\Sigma^*10\Lambda$
- $L/x \neq L/u$

distinguishing *L/y* from *L/u*

- Let $z = 0$
  - $L/y = \{z \in \Sigma^* \mid yz \in L\}$
    - 10
    - 110
    - Σ^*010
  - $L/u = \{z \in \Sigma^* \mid uz \notin L\}$
    - Σ^*100
- $L/y \neq L/u$

The indistinguishability relation I_L

- The indistinguishability relation I_L on Σ^* is defined by

  $x \, I_L \, y$ if and only if $L/x = L/y$
- I_L is an equivalent relation on Σ^*
  - Reflexive
  - Transitive
  - Symmetric
- Notation:
  - $[x]$ denotes the equivalent class containing x
- Two strings are distinguishable with respect to L if they are in different classes of I_L

Properties of I_L

- Any string within the same partition of I_L can be considered the same for operations on strings involving I_L
- I_L is right invariant with respect to concatenation:
  - For $x, y \in \Sigma^*$ and any $a \in \Sigma$,

    if $x \, I_L \, y$ then $xa \, I_L \, ya$

    if $[x] = [y]$ then $[xa] = [ya]$

The algorithm

- Let $L \subseteq \Sigma^*$ and Q_L the set of equivalent classes of I_L on Σ^*. If Q_L is finite then $M_L = (Q_L, \Sigma, q_0, A_L, \delta)$ is FA accepting L, where
  - $q_0 = [\Lambda]$
  - $A_L = \{q \in Q_L \mid q \cap L \neq \Phi\}$
  - δ: $Q_L \times \Sigma \to Q_L$ is

    $\delta([x], a) = [xa]$
- M_L is the FA with the minimum number of states that accepts L
- Extended transition function for M_L:

  $\delta^*([x], y) = [xy]$

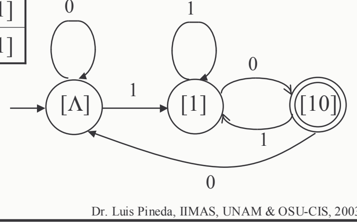
Now we can make sure!

- $M_L = (Q_L, \{0, 1\}, [\Lambda], \{[10]\}, \delta)$ where
  - $Q_L = \{[\Lambda], [1], [10]\}$
    - $[\Lambda] = \{\Lambda, 0, \Sigma^*00\}$
    - $[1] = \{1, 11, \Sigma^*01\}$
    - $[10] = \{\Sigma^*10\}$
- Definition of δ: $\delta([x], a) = [xa]$
  - $\delta([\Lambda], 0) = [\Lambda 0] = [0] = [\Lambda]$ and $\delta([\Lambda], 1) = [\Lambda 1] = [1]$
  - $\delta([1], 0) = [10]$ and $\delta([1], 1) = [11] = [1]$
  - $\delta([10], 0) = [100] = [\Lambda]$ and $\delta([10], 1) = [101] = [1]$
- Notice that you can choose any string in the class as the name of the class: they all are equivalent!

Now we can make sure!

- $M_L = (Q_L, \{0, 1\}, [\Lambda], \{[10]\}, \delta)$ where
  - $Q_L = \{[\Lambda], [1], [10]\}$
  - δ:

Q_L	0	1
[Λ]	[Λ]	[1]
[1]	[10]	[1]
[10]	[Λ]	[1]



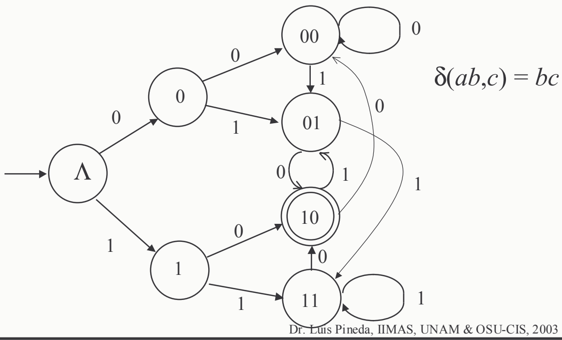
Myhill and Nerode Theorem

- L is a regular language if and only if the set of equivalence classes is finite
- Regularity is a property of a language, not a property of the languages we use to study languages (*RE*) or the FA that accepts the language!
- We are interested in properties of the objects of study… more than in the properties of the physical or abstract devices we use to study these objects!

A related problem

- Finding I_L can be very difficult!
- However, if we have a FA already, can we find its equivalent minimal automata?
- This is more practical for real applications!

Is there an algorithm for finding the minimal FA?



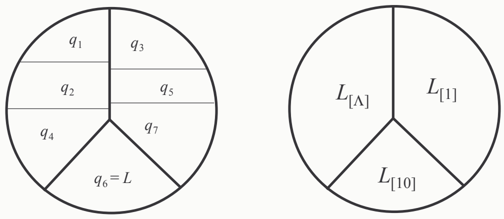
A second way to partition the set of strings!

- The language consisting of the sets of strings that reach a state from the initial state:

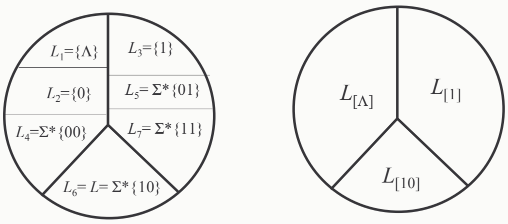
$$L_q = \{x \in \Sigma^* \mid \delta^*(q_0, x) = q\}$$

- The set of all strings is the union of these sets for all $q \in Q$!
- The states of a FA offer a second partition on the set of strings

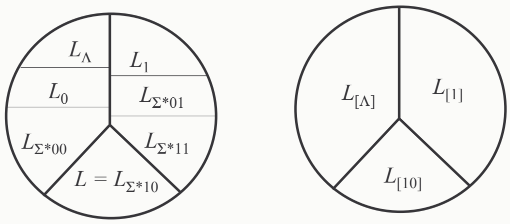
$L_q = \{x \in \Sigma^* \mid \delta^*(q_0, x) = q\}$



$L_q = \{x \in \Sigma^* \mid \delta^*(q_0, x) = q\}$



$L_q = \{x \in \Sigma^* \mid \delta^*(q_0, x) = q\}$



A number of classes determined by the states of a FA will correspond to a single partition in I_L

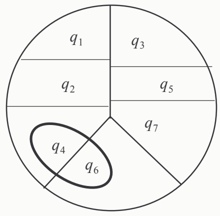
The reduction

- Let (p, q) be pairs of states in the same equivalent class
- The languages L_p and L_q are subsets of the same equivalent class
  - $p \equiv q$
- On the other hand, if p and q do not belong to the same partition, the corresponding languages are subsets of different equivalent classes
  - $p \not\equiv q$
- The minimal FA can be found identifying al (p, q) such that $p \not\equiv q$

The property we need…

S is the set of pairs (p, q) such that $p \not\equiv q$

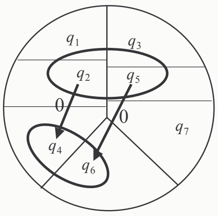
Accepting states and all non-accepting are in different partitions:



$(q_4, q_6) \in S$

The property we need…

S is the set of pairs (p, q) such that $p \not\equiv q$



$(q_4, q_6) \in S$

and

(q_2, q_5) is such that

$\delta(q_2, 0) = q_4$

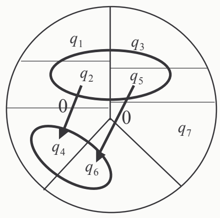
$\delta(q_5, 0) = q_6$

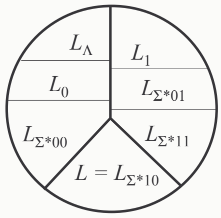
then $(q_2, q_5) \in S$.

0 is a distinguishing string for states q_2 and q_5

The property we need…

- 0 is a distinguishing string taken strings in L_{Σ^*01} into L but not strings in L_0 into L
- Subsequently, q_2 and q_5 belong to different partitions of I_L





The property we need…

- If (p, q) is a pair of states such that $p \not\equiv q$ and $\delta(r, a) = p$ and $\delta(s, a) = q$, then (r, s) is also a pair of states such that $r \not\equiv s$
- If $p \not\equiv q$ then there is a z such that
  - $\delta^*(p, z) \in A$ or $\delta^*(q, z) \in A$, but not both
- So, only one of
  - $\delta^*(r, az) = \delta^*(\delta(r, a), z) = \delta^*(p, z)$
  - $\delta^*(s, az) = \delta^*(\delta(s, a), z) = \delta^*(q, z)$

  will be in A!
- Paths of non-equivalent states are formed by non-equivalent states!

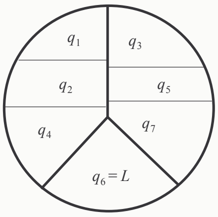
The algorithm…

- Let S be the set of pairs (p, q) such that $p \not\equiv q$
  - For any p and q such that only one of them is in A, include (p, q) in S
  - For any pair $(p, q) \in S$, if (r, s) is a pair for which $\delta(r, a) = p$ and $\delta(s, a) = q$ then $(r, s) \in S$
  - No other pairs are in S

Compute *S*

S is the set of pairs (p, q) such that $p \not\equiv q$

1. For any p and q such that only one of them is in *A,* include (p, q) in *S*
2. So far, we only know that the accepting state is not in the same class of non-accepting states



$S = \{(q_1, q_6), (q_2, q_6), (q_3, q_6), (q_4, q_6), (q_5, q_6), (q_7, q_6)\}$

Compute *S*

S is the set of pairs (p, q) such that $p \not\equiv q$

2. For any pair $(p, q) \in S$, if (r, s) is such that
   $\delta(r, a) = p$
   $\delta(s, a) = q$
   then $(r, s) \in S$.

- Note that *S* is also a relation (the one of states in different partitions)
- The complement of this relation, will be also a relation, but the one of the states in the same partition!

Computation of *S*

q_7						1	
q_6						0	
q_5						1	
q_4						1	
q_3						1	
q_2						1	
q_1						1	
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

$S = \{(q_6, q_1), (q_6, q_2), (q_6, q_3), (q_6, q_4), (q_6, q_5), (q_6, q_7)\}$

Computation of *S*

q_7						1	
q_6	1	1	1	1	1	0	1
q_5						1	
q_4						1	
q_3						1	
q_2						1	
q_1						1	
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

S is symmetrical!

Computation of *S*

q_7						1	0
q_6	1	1	1	1	1	0	1
q_5					0	1	
q_4				0		1	
q_3			0			1	
q_2		0				1	
q_1	0					1	
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

We also know that no state is related to itself in *S*

Computation of *S*

q_7						1	0
q_6	1	1	1	1	1	0	1
q_5					0	1	
q_4				0		1	
q_3			0			1	
q_2		0				1	
q_1	0					1	
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

Finding the pairs (r, s)

If (r, s) is such that $\delta(r, a) = p$ and $\delta(s, a) = q$ then $(r, s) \in S$.

Computation of *S*

q_7						1	0
q_6	1	1	1	1	1	0	1
q_5					0	1	
q_4				0		1	
q_3			0			1	
q_2		0				1	
q_1	0					1	
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

Q	0	1
q_1	q_2	q_1
q_2	q_4	q_5
q_3	q_6	q_7
q_4	q_4	q_5
q_5	q_6	q_7
q_6	q_4	q_5
q_7	q_6	q_7

To compute (r, s) we need to look at δ going backwards

Computation of *S*: going backwards

q_7						1	0
q_6	1	1	1	1	1	0	1
q_5					0	1	
q_4				0		1	
q_3			0			1	
q_2		0				1	
q_1	0					1	
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

Q	0	1
q_1	q_2	q_1
q_2	q_4	q_5
q_3	q_6	q_7
q_4	q_4	q_5
q_5	q_6	q_7
q_6	q_4	q_5
q_7	q_6	q_7

If (r, s) is such that $\delta(r, a) = q_4$ and $\delta(s, a) = q_6$

then $(r, s) \in S$.

Computation of *S*: going backwards

q_7						1	0
q_6	1	1	1	1	1	0	1
q_5		1			0	1	
q_4				0		1	
q_3			0			1	
q_2		0				1	
q_1	0					1	
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

Q	0	1
q_1	q_2	q_1
q_2	q_4	q_5
q_3	q_6	q_7
q_4	q_4	q_5
q_5	q_6	q_7
q_6	q_4	q_5
q_7	q_6	q_7

(q_2, q_5) is one such pair:

$\delta(q_2, 0) = q_4$ and $\delta(q_5, 0) = q_6$ then $(q_2, q_5) \in S$.

Computation of *S*: going backwards

q_7						1	0
q_6	1	1	1	1	1	0	1
q_5		1			0	1	
q_4				0		1	
q_3			0			1	
q_2		0			1	1	
q_1	0					1	
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

S is symmetrical: $(q_5, q_2) \in S$.

Computation of *S*: going backwards

q_7						1	0
q_6	1	1	1	1	1	0	1
q_5		1			0	1	
q_4				0		1	
q_3			0			1	
q_2		0				1	
q_1	0					1	
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

S is symmetrical: We only need to find the values for the relation in one way

Computation of *S*

q_7	1	1		1		1	0
q_6	1	1	1	1	1	0	1
q_5	1	1		1	0	1	
q_4			1	0		1	
q_3	1	1	0			1	
q_2		0				1	
q_1	0					1	
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

Q	0	1
q_1	q_2	q_1
q_2	q_4	q_5
q_3	q_6	q_7
q_4	q_4	q_5
q_5	q_6	q_7
q_6	q_4	q_5
q_7	q_6	q_7

The pairs found by searching backwards!

Computation of *S*

q_7	1	1		1		1	0
q_6	1	1	1	1	1	0	1
q_5	1	1		1	0	1	
q_4			1	0	1	1	1
q_3	1	1	0	1		1	
q_2		0	1		1	1	1
q_1	0		1		1	1	1
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

S is symmetrical!

Computation of *S*

q_7	1	1	0	1	0	1	0
q_6	1	1	1	1	1	0	1
q_5	1	1	0	1	0	1	0
q_4	0	0	1	0	1	1	1
q_3	1	1	0	1	0	1	0
q_2	0	0	1	0	1	1	1
q_1	0	0	1	0	1	1	1
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

The values for all the matrix of states in different classes

The relation of related states:

q_7			1		1		1
q_6						1	
q_5			1		1		1
q_4	1	1		1			
q_3			1		1		1
q_2	1	1		1			
q_1	1	1		1			
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

The complement: the relation between the pairs in the **same** partition**!**

Computation of Q_L

q_7			1		1		1
q_6						1	
q_5			1		1		1
q_4	1	1		1			
q_3			1		1		1
q_2	1	1		1			
q_1	1	1		1			
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

The relation is symmetrical

Computation of Q_L

q_7			1		1		1
q_6						1	
q_5			1		1		1
q_4	1	1		1			
q_3			1		1		1
q_2	1	1		1			
q_1	1	1		1			
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

The relation is reflexive

Computation of Q_L

q_7			1		1		1
q_6						1	
q_5			1		1		1
q_4	1	1		1			
q_3			1		1		1
q_2	1	1		1			
q_1	1	1		1			
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

The relation is transitive: States q_1, q_2 and q_4 are mutually related, but not with any one else

Computation of Q_L

q_7			1		1		1
q_6						1	
q_5			1		1		1
q_4	1	1		1			
q_3			1		1		1
q_2	1	1		1			
q_1	1	1		1			
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

The relation is transitive: States q_3, q_5 and q_7 are mutually related, but not with any one else

Computation of Q_L

q_7			1		1		1
q_6						1	
q_5			1		1		1
q_4	1	1		1			
q_3			1		1		1
q_2	1	1		1			
q_1	1	1		1			
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

The relation is transitive: State q_6 is related with itself but not with any one else!

Computation of Q_L

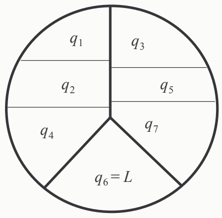
q_7					1	1	1
q_5					1	1	1
q_3					1	1	1
q_6				1			
q_4	1	1	1				
q_2	1	1	1				
q_1	1	1	1				
	q_1	q_2	q_4	q_6	q_3	q_5	q_7

Reordering the set of states!

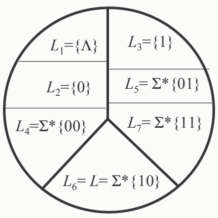
The relation is transitive: the classes form clusters where the properties of the reflexive relation can be seen directly!

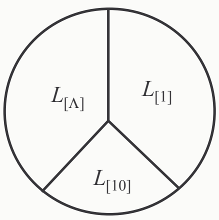
The set of equivalent states!

q_7					1	1	1
q_5					1	1	1
q_3					1	1	1
q_6				1			
q_4	1	1	1				
q_2	1	1	1				
q_1	1	1	1				
	q_1	q_2	q_4	q_6	q_3	q_5	q_7



The minimal set of states: Q_L





The reduction is complete!

Computation of S

q_7	1	1		1		1	0
q_6	1	1	1	1	1	0	1
q_5	1	1		1	0	1	
q_4			1	0		1	
q_3	1	1	0			1	
q_2		0				1	
q_1	0					1	
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

Mark the pairs found by searching backwards in one or more passes

The pairs found by searching backwards in the upper left part of the matrix!

The algorithm

q_7	1	1		1		1	
q_6	1	1	1	1	1		
q_5	1	1		1			
q_4			1				
q_3	1	1					
q_2							
q_1							
	q_1	q_2	q_3	q_4	q_5	q_6	q_7

- The pairs that are not marked belong to equivalent classes
- Add the pairs relating the accepting states with themselves

[Λ] = {(1, 2), (1,4), (2,4)}
[1] = {(3, 5), (3, 7), (5, 7)}
[10] = {(6, 6)} (The reference state)

The algorithm

1. Include in S all pairs containing only one accepting state
2. Select region of matrix: if $(x, y) \in S$ and $y < x$ add (y, x) in S (as S is symmetrical)
3. Define Q as the set of pairs of the upper left part of the matrix that are not in S
4. For each $(x, y) \in Q$, check whether it can reach a pair in S in one transition
   – If it does, remove (x, y) from Q and include it in S
5. Repeat 4 until no more pairs can be added to S
6. Add to Q all pairs (x, x), where x is an accepting state
7. Q is the equivalence relation sought

Example

- Find the minimal automata M_L for the FA M:

M

Q	a	b
→1	2	1
*2	4	1
3	2	5
4	6	3
*5	4	5
6	5	2

S

6		1			1	0
5	1	0	1	1	0	1
4		1		0	1	
3		1	0		1	
2	1	0	1	1	0	1
1	0	1			1	
Q	1	2	3	4	5	6

- S is the relation of between states of different partitions
- Initial assumption for S: (p, q) such that *exactly* one element is in A

Example … cont

(r, s)	$\delta(r, a)$	$\delta(s, a)$	(p, q)	Pass
(4, 6)	6	5	(6, 5)	1
(3, 6)	2	5	(2, 5)	
(3, 4)	2	6	(2, 6)	1
(1, 6)	2	5	(2, 5)	
(1, 4)	2	6	(2, 6)	1
(1, 3)	2	2	(2, 2)	

S

6	0	1	0	1	1	0
5	1	0	1	1	0	1
4	1	1	1	0	1	1
3	0	1	0	1	1	0
2	1	0	1	1	0	1
1	0	1	0	1	1	0
Q	1	2	3	4	5	6

- The related states are: [1] = {1, 3, 6}, [4] and [2] = {2, 5}

Example … cont

S

6	0	1	0	1	1	0
5	1	0	1	1	0	1
4	1	1	1	0	1	1
3	0	1	0	1	1	0
2	1	0	1	1	0	1
1	0	1	0	1	1	0
Q	1	2	3	4	5	6

E

6	1		1			1
5		1			1	
4				1		
3	1		1			1
2		1			1	
1	1		1			1
Q	1	2	3	4	5	6

E

5					1	1
2					1	1
4				1		
6	1	1	1			
3	1	1	1			
1	1	1	1			
Q	1	3	6	4	2	5

- The equivalent relation: [1] = {1, 3, 6}, [2] = {2, 5} and [4]

Example … cont

- $M_L = (Q_L, \{0, 1\}, [136], \{[25]\}, \delta)$ where
- Q = {[136], [25], [4]},
- δ:
  – δ([136], a) = [25] and δ([136], b) = [15]?
  – δ([25], a) = [4] and δ([25], b) = [1] = [15]?
  – δ([4], a) = [6] and δ([4], b) = [3] = [136]
- The string a is not distinguishing if the final states are equivalent!

Q	a	b
→1	2	1
*2	4	1
3	2	5
4	6	3
*5	4	5
6	5	2

Example … cont

- Assume accepting states are not related!

M

Q	a	b
→1	2	1
*2	4	1
3	2	5
4	6	3
*5	4	5
6	5	2

S

6		1			1	0
5	1	1	1	1	0	1
4		1		0	1	
3		1	0		1	
2	1	0	1	1	1	1
1	0	1			1	
Q	1	2	3	4	5	6

- S is the relation of between states of different partitions

Example … cont

(r, s)	$\delta(r, a)$	$\delta(s, a)$	(p, q)	Pass
(4, 6)	6	5	(6, 5)	1
(3, 6)	2	5	(2, 5)	
(3, 4)	2	6	(2, 6)	1
(1, 6)	2	5	(2, 5)	
(1, 4)	2	6	(2, 6)	1
(1, 3)	2	2	(2, 2)	

S

6	1	1	1	1	1	0
5	1	1	1	1	0	1
4	1	1	1	0	1	1
3	0	1	0	1	1	0
2	1	0	1	1	1	1
1	0	1	0	1	1	0
Q	1	2	3	4	5	6

- The related states are: [1] = {1, 3}, [2], [4], [5] and [6]

Example … cont

S

6	1	1	1	1	1	0
5	1	1	1	1	0	1
4	1	1	1	0	1	1
3	0	1	0	1	1	1
2	1	0	1	1	1	1
1	0	1	0	1	1	1
Q	1	2	3	4	5	6

E

6						1
5					1	
4				1		
3	1		1			
2		1				
1	1		1			
Q	1	2	3	4	5	6

E

6						1
5					1	
4				1		
2			1			
3	1	1				
1	1	1				
Q	1	3	2	4	5	6

- The equivalent relation: [1] = {1, 3}, [2], [4], [5] and [6]

Example … cont

- $M_L = (Q_L, \{0, 1\}, [136], \{[25]\}, \delta)$ where
- Q = {[13], [2], [4], [5], [6]}
- δ:
  - δ([13], a) = [2] and δ([13], b) = [15]?
  - δ([25], a) = [4] and δ([25], b) = [1] = [15]?
  - δ([4], a) = [6] and δ([4], b) = [3] = [136]
- 1 and 5 are in different partitions: a is not distinguishing!

Q	a	b
→1	2	1
*2	4	1
3	2	5
4	6	3
*5	4	5
6	5	2

Example … cont

- We take b to be the distinguishing string, and the accepting states to be related

(r, s)	$\delta(r, b)$	$\delta(s, b)$	(p, q)	Pass
(4, 6)	3	2	(3, 2)	1
(3, 6)	5	2	(5, 2)	
(3, 4)	5	3	(5, 3)	1
(1, 6)	1	2	(1, 2)	1
(1, 4)	1	3	(1, 3)	2
(1, 3)	1	5	(1, 5)	1

S

6	1	1	0	1	1	0
5	1	0	1	1	0	1
4	1	1	1	0	1	1
3	1	1	0	1	1	0
2	1	0	1	1	0	1
1	0	1	0	1	1	1
Q	1	2	3	4	5	6

- The related states are: [1], [3] = {3, 6}, [4] and [2] = {2, 5}

Example … cont

- $M_L = (Q_L, \{0, 1\}, [1], \{[25]\}, \delta)$ where
- Q = {[1], [25], [36], [4]},
- δ:
  - δ([1], a) = [2] = [25] and δ([1], b) = [1]
  - δ([25], a) = [4] and δ([25], b) = [1] = [15] ?
  - δ([4], a) = [6] and δ([4], b) = [3] = [136]
- 1 and 5 belong to different partitions, so the accepting states cannot be related

Q	a	b
→1	2	1
*2	4	1
3	2	5
4	6	3
*5	4	5
6	5	2

Example … cont

- We take *b* to be the distinguishing string, and the accepting states NOT to be related

(*r*, *s*)	δ(*r*, *b*)	δ(*s*, *b*)	(*p*, *q*)	Pass
(4, 6)	3	2	(3, 2)	1
(3, 6)	5	2	(5, 2)	1
(3, 4)	5	3	(5, 3)	1
(1, 6)	1	2	(1, 2)	1
(1, 4)	1	3	(1, 3)	2
(1, 3)	1	5	(1, 5)	1

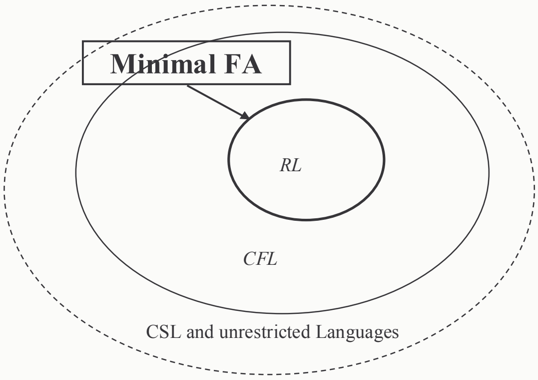
S						
6	1	1	1	1	1	0
5	1	1	1	1	0	1
4	1	1	1	0	1	
3	1	1	0		1	
2	1	0	1	1	1	1
1	0	1			1	
Q	1	2	3	4	5	6

- There are no equivalent states and the FA is already minimal

Minimal Automata

- For every FA there is always a minimal FA with *n* states
- This is useful for simplification of FA
- Minimal FA also provide a "normal form" for FA: all FA that reduce to the minimal one are equivalent
- This also provides a test for equivalence between FA!
- Using Kleen's Theorem we can find the "normal form" for a *RE*

Normal form for *RE* and FA



The power of *RE* and FA

- A *problem* is the question of deciding whether a string is a member of some particular language
- FA solve simple decision problems (i.e. whether *x* is an even number, but not whether *x* is a palindrome).
- Generic decision problem for a FA: given a strings *x* and a FA *M* decide whether *x* is accepted by *M*:

(x, *M*) → Decision Process for *RL* → Yes/no

Decision problems for *RL*

- Given a *RE* r and a string x, $x \subseteq L(r)$?
- Given a FA M, is $L(M) = \Phi$?
- Given a FA M, is $L(M)$ finite?
- Given M_1 and M_2, is $L(M_1) \cap L(M_2) \neq \Phi$?
- Given M_1 and M_2, is $L(M_1) = L(M_2)$?
- Given M_1 and M_2, is $L(M_1) \subseteq L(M_2)$?
- Given two *RE*s r_1 and r_2, is $L(r_1) = L(r_2)$?
- Given a FA M is it minimal?

RL is a very solid system for simple problems!