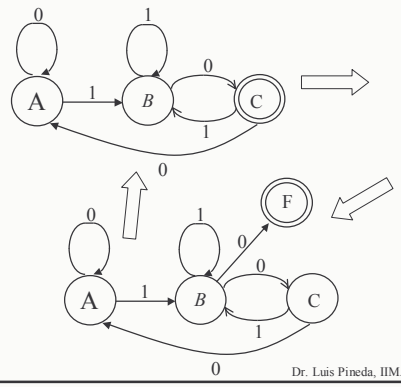
Session 21

CFG Corresponding to a PDA

PDA and CFG

- ✓ There is a PDA M such that $L(M) = L(G)$ for every CFG G
- There is a CFG G such that $L(G) = L(M)$ for every PDA M
- The set of *CFL* generated by CFG (ambiguous or unambiguous) is the set of *CFL* accepted by PDA.

An analogy with FA!



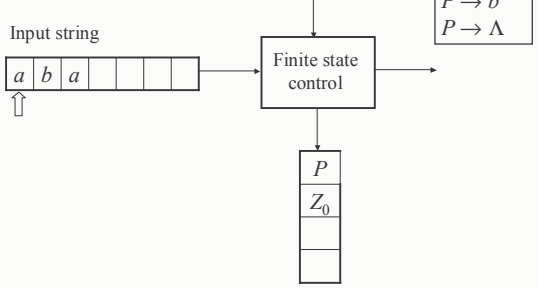
$A \to 0A$
$A \to 1B$
$B \to 0C \mid 0$
$B \to 1B$
$C \to 0A$
$C \to 1B$

A Top-down simulation of *Pal*

$M = (Q, \Sigma, \Gamma, q_0, Z_0, A, \delta)$

The productions

$P \to aPa$
$P \to bPb$
$P \to a$
$P \to b$
$P \to \Lambda$



Top-down process of *Pal* = *aba*

string	state	input	Stack	type: move	Conf.	Production
aba	q_0	Λ	Z_0	0: (q_1, P)	(q_0, aba, Z_0)	
aba	q_1	Λ	P	1: (q_1, aPa)	(q_1, aba, PZ_0)	$P \to aPa$
aba	q_1	a	a	2: (q_1, Λ)	$(q_1, aba, aPaZ_0)$	
ba	q_1	Λ	P	1: (q_1, b)	(q_1, ba, PaZ_0)	$P \to b$
ba	q_1	b	b	2: (q_1, Λ)	(q_1, ba, baZ_0)	
a	q_1	a	a	2: (q_1, Λ)	(q_1, a, aZ_0)	
Λ	q_1	Λ	Z_0	3: (q_2, Z_0)	(q_1, Λ, Z_0)	
Λ	q_2	Λ	Z_0		(q_2, Λ, Z_0)	

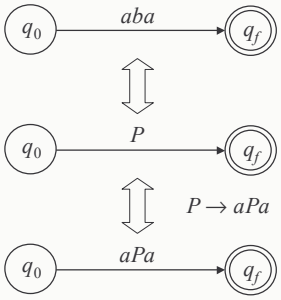
Productions correspond to moves of type 1

The grammar of the PDA!

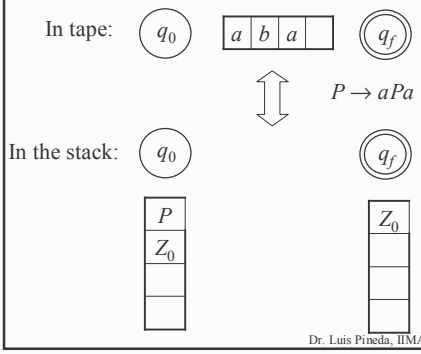
string	state	input	Stack	type: move	Conf.	Production
aba	q_0	Λ	Z_0	0: (q_1, P)	(q_0, aba, Z_0)	
aba	q_1	Λ	P	1: (q_1, aPa)	(q_1, aba, PZ_0)	$P \to aPa$
aba	q_1	a	a	2: (q_1, Λ)	$(q_1, aba, aPaZ_0)$	
ba	q_1	Λ	P	1: (q_1, b)	(q_1, ba, PaZ_0)	$P \to b$
ba	q_1	b	b	2: (q_1, Λ)	(q_1, ba, baZ_0)	
a	q_1	a	a	2: (q_1, Λ)	(q_1, a, aZ_0)	
Λ	q_1	Λ	Z_0	3: (q_2, Z_0)	(q_1, Λ, Z_0)	
Λ	q_2	Λ	Z_0		(q_2, Λ, Z_0)	

Is it possible to associate a production to each move?

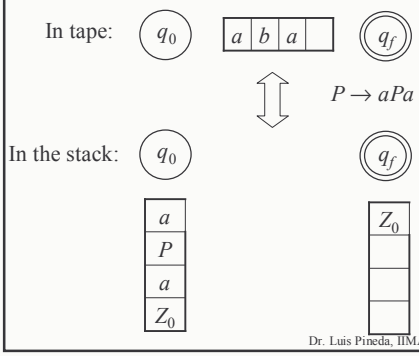
Correspondence between states and production



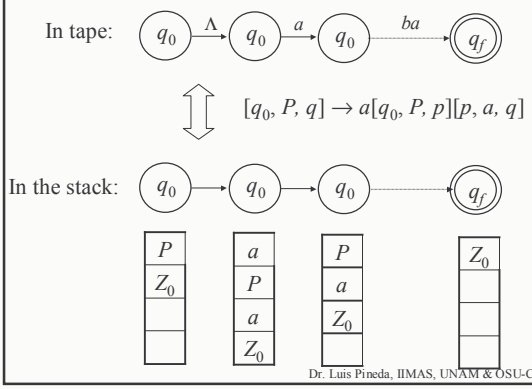
Correspondence between states and production



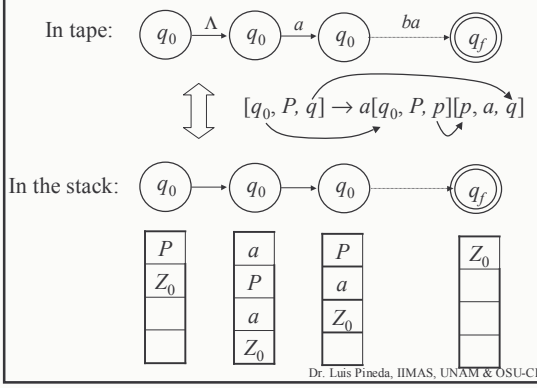
Correspondence between states and production



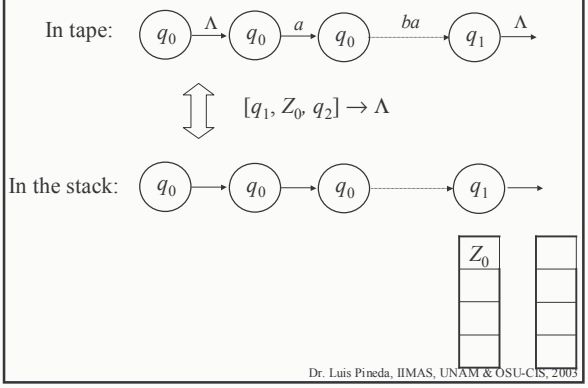
States and production



States and production



The initial stack symbol is like any variable!



Finding a Grammar for *a* PDA

- Convert the PDA accepting by final state to the corresponding PDA accepting by empty stack
- Provide the set of variable symbols (i.e. Γ) :
  - Symbols correspond to variables in the grammar
  - Go into the stack in moves type 1 (i.e. pop symbol on top of the stack and push right side of production)
  - Match symbols of the input string in Σ in moves of type 2 (i.e. consume input and pop)
- Find the productions that correspond to each state and move of the PDA. These productions have the form $A \rightarrow a\alpha$ where
  - A is the top of the stack
  - a is the symbol in the input string
  - α is the string replacing the top of the stack by the move

Transition table of Pal_{mark}

Id	State	Input	Stack symbol	Move
1	q_0	a	Z_0	(q_0, aZ_0)
2	q_0	b	Z_0	(q_0, bZ_0)
3	q_0	a	a	(q_0, aa)
4	q_0	b	a	(q_0, ba)
5	q_0	a	b	(q_0, ab)
6	q_0	b	b	(q_0, bb)
7	q_0	c	Z_0	(q_1, Z_0)
8	q_0	c	a	(q_1, a)
9	q_0	c	b	(q_1, b)
10	q_1	a	a	(q_1, Λ)
11	q_1	b	b	(q_1, Λ)
12	q_1	Λ	Z_0	(q_2, Z_0)
Other combinations				non

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Accepting by empty stack

Id	State	Input	Stack symbol	Move
1	q_0	a	Z_0	(q_0, aZ_0)
2	q_0	b	Z_0	(q_0, bZ_0)
3	q_0	a	a	(q_0, aa)
4	q_0	b	a	(q_0, ba)
5	q_0	a	b	(q_0, ab)
6	q_0	b	b	(q_0, bb)
7	q_0	c	Z_0	(q_1, Z_0)
8	q_0	c	a	(q_1, a)
9	q_0	c	b	(q_1, b)
10	q_1	a	a	(q_1, Λ)
11	q_1	b	b	(q_1, Λ)
12	q_1	Λ	Z_0	(q_1, Λ)
Other combinations				non

In push moves variables count symbols!

Id	State	Input	Stack symbol	Move
1	q_0	a	Z_0	(q_0, AZ_0)
2	q_0	b	Z_0	(q_0, BZ_0)
3	q_0	a	A	(q_0, AA)
4	q_0	b	A	(q_0, BA)
5	q_0	a	B	(q_0, AB)
6	q_0	b	B	(q_0, BB)
7	q_0	c	Z_0	(q_1, Z_0)
8	q_0	c	A	(q_1, A)
9	q_0	c	B	(q_1, B)
10	q_1	a	A	(q_1, Λ)
11	q_1	b	B	(q_1, Λ)
12	q_1	Λ	Z_0	(q_1, Λ)
Other combinations				non

The job of the productions

- For each symbols that is pushed in the first half of the string, there must be a symbol to cancel it with the corresponding symbol in the second half!
- The low of symbols preservation in *Pal*: symbols are never created or destroyed, they just cancel each other!
- The rule to move hypothesis: for each move there is a production!

The grammar: first version

Id	State	Input	Stack	Move	Productions
1	q_0	a	Z_0	(q_0, AZ_0)	$Z_0 \rightarrow aAZ_0$
2	q_0	b	Z_0	(q_0, BZ_0)	$Z_0 \rightarrow bBZ_0$
3	q_0	a	A	(q_0, AA)	$A \rightarrow aAA$
4	q_0	b	A	(q_0, BA)	$A \rightarrow bBA$
5	q_0	a	B	(q_0, AB)	$B \rightarrow aAB$
6	q_0	b	B	(q_0, BB)	$B \rightarrow bBB$
7	q_0	c	Z_0	(q_1, Z_0)	$Z_0 \rightarrow cZ_0$
8	q_0	c	A	(q_1, A)	$A \rightarrow cA$
9	q_0	c	B	(q_1, B)	$B \rightarrow cB$
10	q_1	a	A	(q_1, Λ)	$A \rightarrow a\Lambda$
11	q_1	b	B	(q_1, Λ)	$B \rightarrow b\Lambda$
12	q_1	Λ	Z_0	(q_1, Λ)	$Z_0 \rightarrow \Lambda$
Other combinations				non	

Top-down simulation

- We consider some typical moves:

Id	State	Input	Stack symbol	Move
1	q_0	a	Z_0	(q_0, AZ_0)
3	q_0	a	A	(q_0, AA)

- If we see an a (or a b) in the first half of the string, we push a variable matching such input, such that we will be able to pop the corresponding a in the second part!
- Conversely, to generate such a in the first part of the string and its corresponding a in the second we need rules of the form:

$$Z_0 \to aAZ_0 \text{ and } A \to aAA$$

Top-down simulation

- We consider some typical moves:

Id	State	Input	Stack symbol	Move
7	q_0	c	Z_0	(q_1, Z_0)
8	q_0	c	A	(q_1, A)

- If we see a c, we need to consume it and change state to process the second part of the string!
- Conversely, to generate such c we need rules of the form:

$$Z_0 \to cZ_0 \text{ and } A \to cA$$

Top-down simulation

- We consider some typical moves:

Id	State	Input	Stack symbol	Move
10	q_1	a	A	(q_1, Λ)

- If we see an a (or a b) in the second part, we need to pop it
- Conversely, to generate terminal a's we need a rule of the form:

$$A \to a \qquad \text{(i.e. } A \to a\Lambda\text{)}$$

Top-down simulation

- We consider some typical moves:

Id	State	Input	Stack symbol	Move
12	q_1	Λ	Z_0	(q_1, Λ)

- If we see Λ in the second part, we need to accept (by empty stack!)
- Conversely, to generate such Λ in the second half we need a rule of the form:

$$Z_0 \to \Lambda \qquad \text{(i.e. } Z_0 \to \Lambda\Lambda\text{)}$$

A leftmost derivation

- Derive aca:

$$\begin{aligned} Z_0 &\Rightarrow aAZ_0 && \text{(by 1: } Z_0 \to aAZ_0\text{)} \\ &\Rightarrow acAZ_0 && \text{(by 8: } A \to cA\text{)} \\ &\Rightarrow acaZ_0 && \text{(by 10: } A \to a\Lambda\text{)} \\ &\Rightarrow aca\Lambda = aca && \text{(by 12: } Z_0 \to \Lambda\text{)} \end{aligned}$$

A leftmost derivation

- However this grammar over-generates:

$$\begin{aligned} Z_0 &\Rightarrow aAZ_0 && \text{(by 1: } Z_0 \to aAZ_0\text{)} \\ &\Rightarrow aaZ_0 && \text{(by 10: } A \to a\Lambda\text{)} \\ &\Rightarrow aa\Lambda && \text{(by 12: } Z_0 \to \Lambda\text{)} \end{aligned}$$

  – but aa is not in this language!

  – Despite that aa is not accepted by the PDA:

$$(q_0, aa, Z_0) \Rightarrow (q_0, a, AZ_0) \Rightarrow (q_0, \Lambda, AAZ_0)$$

- This grammar is missing something!

Preventing overgeneration

- Production rules are "bound" to states
  - Rule 1: Transition from state q_0 back to q_0
  - Rule 10: Transition from state q_1 back to q_1
  - Transition from q_0 to q_1: moves 7, 8 or 9

    $Z_0 \Rightarrow aAZ_0$ (by 1)

    … (eventually 7, 8 or 9)

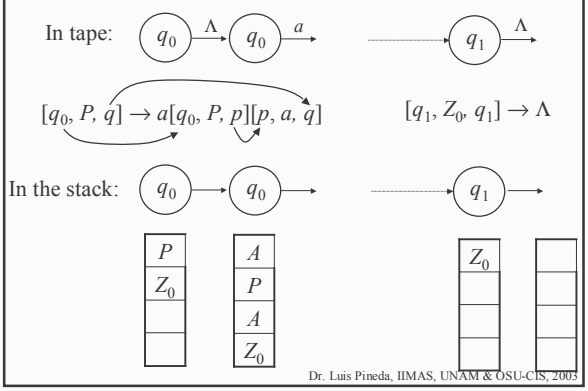
    $\Rightarrow aaZ_0$ (by 10)

    $\Rightarrow aa\Lambda$ (by 12)
  - To use production 10, one of productions 7, 8 or 9 must have been used before in the derivation!
- To block over generation we need to codify in the productions of the grammar the corresponding states of the PDA!

The rules of the over-generation case!

Id	State	Input	Stack	Move	Productions
1	q_0	a	Z_0	(q_0, AZ_0)	$Z_0 \rightarrow aAZ_0$
2	q_0	b	Z_0	(q_0, BZ_0)	$Z_0 \rightarrow bBZ_0$
3	q_0	a	A	(q_0, AA)	$A \rightarrow aAA$
4	q_0	b	A	(q_0, BA)	$A \rightarrow bBA$
5	q_0	a	B	(q_0, AB)	$B \rightarrow aAB$
6	q_0	b	B	(q_0, BB)	$B \rightarrow bBB$
7	q_0	c	Z_0	(q_1, Z_0)	$Z_0 \rightarrow cZ_0$
8	q_0	c	A	(q_1, A)	$A \rightarrow cA$
9	q_0	c	B	(q_1, B)	$B \rightarrow cB$
10	q_1	a	A	(q_1, Λ)	$A \rightarrow a$
11	q_1	b	B	(q_1, Λ)	$B \rightarrow b$
12	q_1	Λ	Z_0	(q_1, Λ)	$Z_0 \rightarrow \Lambda$
Other combinations				non	

Codifying the states into the variables!



The grammar

Id	State	Input	Stack	Move	Productions
0					$S \rightarrow [q_0, Z_0, q]$
1	q_0	a	Z_0	(q_0, AZ_0)	$[q_0, Z_0, q] \rightarrow a[q_0, A, p][p, Z_0, q]$
2	q_0	b	Z_0	(q_0, BZ_0)	$[q_0, Z_0, q] \rightarrow b[q_0, B, p][p, Z_0, q]$
3	q_0	a	A	(q_0, AA)	$[q_0, A, q] \rightarrow a[q_0, A, p][p, A, q]$
4	q_0	b	A	(q_0, BA)	$[q_0, A, q] \rightarrow b[q_0, B, p][p, A, q]$
5	q_0	a	B	(q_0, AB)	$[q_0, B, q] \rightarrow a[q_0, A, p][p, B, q]$
6	q_0	b	B	(q_0, BB)	$[q_0, B, q] \rightarrow b[q_0, B, p][p, B, q]$
7	q_0	c	Z_0	(q_1, Z_0)	$[q_0, Z_0, q] \rightarrow c[q_1, Z_0, q]$
8	q_0	c	A	(q_1, A)	$[q_0, A, q] \rightarrow c[q_1, A, q]$
9	q_0	c	B	(q_1, B)	$[q_0, B, q] \rightarrow c[q_1, B, q]$
10	q_1	a	A	(q_1, Λ)	$[q_1, A, q_1] \rightarrow a$
11	q_1	b	B	(q_1, Λ)	$[q_1, B, q_1] \rightarrow b$
12	q_1	Λ	Z_0	(q_1, Λ)	$[q_1, Z_0, q_1] \rightarrow \Lambda$
Other combinations				non	

Accepting and generating *bacab*

- The moves:

$(q_0, bacab, Z_0) \Rightarrow (q_0, acab, BZ_0)$

$\Rightarrow (q_0, cab, ABZ_0)$

$\Rightarrow (q_1, ab, ABZ_0)$

$\Rightarrow (q_1, b, BZ_0)$

$\Rightarrow (q_1, \Lambda, Z_0)$

$\Rightarrow (q_1, \Lambda, \Lambda)$

Accepting and generating *bacab*

- The derivation:

$S \Rightarrow [q_0, Z_0, q]$ (0)

Accepting and generating *bacab*

- The derivation:

$S \Rightarrow [q_0, Z_0, q]$ (0)
$\Rightarrow b[q_0, B, p][p, Z_0, q]$ (2)

Production 2: $[q_0, Z_0, q] \rightarrow b[q_0, B, p][p, Z_0, q]$

Accepting and generating *bacab*

- The derivation:

$S \Rightarrow [q_0, Z_0, q]$ (0)
$\Rightarrow b[q_0, B, p][p, Z_0, q]$ (2)
$\Rightarrow ba[q_0, A, r][r, B, p][p, Z_0, q]$ (5)

Production 5: $[q_0, B, q] \rightarrow a[q_0, A, p][p, B, q]$

Accepting and generating *bacab*

- The derivation:

$S \Rightarrow [q_0, Z_0, q]$ (0)
$\Rightarrow b[q_0, B, p][p, Z_0, q]$ (2)
$\Rightarrow ba[q_0, A, r][r, B, p][p, Z_0, q]$ (5)
$\Rightarrow bac[q_1, A, r][r, B, p][p, Z_0, q]$ (8)

Production 8: $[q_0, A, q] \rightarrow c[q_1, A, q]$

Accepting and generating *bacab*

- The derivation:

$S \Rightarrow [q_0, Z_0, q]$ (0)
$\Rightarrow b[q_0, B, p][p, Z_0, q]$ (2)
$\Rightarrow ba[q_0, A, r][r, B, p][p, Z_0, q]$ (5)
$\Rightarrow bac[q_1, A, r][r, B, p][p, Z_0, q]$ (8)
$\Rightarrow baca[r, B, p][p, Z_0, q]$ (10)

Production 10: $[q_1, A, q_1] \rightarrow a$

Accepting and generating *bacab*

- The derivation:

$S \Rightarrow [q_0, Z_0, q]$ (0)
$\Rightarrow b[q_0, B, p][p, Z_0, q]$ (2)
$\Rightarrow ba[q_0, A, q_1][q_1, B, p][p, Z_0, q]$ (5)
$\Rightarrow bac[q_1, A, q_1][q_1, B, p][p, Z_0, q]$ (8)
$\Rightarrow baca[q_1, B, p][p, Z_0, q]$ (10)

Production 10: r gets bound to q_1

Accepting and generating *bacab*

- The derivation:

$S \Rightarrow [q_0, Z_0, q]$ (0)
$\Rightarrow b[q_0, B, p][p, Z_0, q]$ (2)
$\Rightarrow ba[q_0, A, q_1][q_1, B, p][p, Z_0, q]$ (5)
$\Rightarrow bac[q_1, A, q_1][q_1, B, p][p, Z_0, q]$ (8)
$\Rightarrow baca[q_1, B, p][p, Z_0, q]$ (10)
$\Rightarrow bacab[p, Z_0, q]$ (11)

Production 11: $[q_1, B, q_1] \rightarrow b$

Accepting and generating *bacab*

- The derivation:

$S \Rightarrow [q_0, Z_0, q]$ (0)
$\Rightarrow b[q_0, B, q_1][q_1, Z_0, q]$ (2)
$\Rightarrow ba[q_0, A, q_1][q_1, B, q_1][q_1, Z_0, q]$ (5)
$\Rightarrow bac[q_1, A, q_1][q_1, B, q_1][q_1, Z_0, q]$ (8)
$\Rightarrow baca[q_1, B, q_1][q_1, Z_0, q]$ (10)
$\Rightarrow bacab[q_1, Z_0, q]$ (11)

Production 11: p gets bound to q_1

Accepting and generating *bacab*

- The derivation:

$S \Rightarrow [q_0, Z_0, q]$ (0)
$\Rightarrow b[q_0, B, q_1][q_1, Z_0, q]$ (2)
$\Rightarrow ba[q_0, A, q_1][q_1, B, q_1][q_1, Z_0, q]$ (5)
$\Rightarrow bac[q_1, A, q_1][q_1, B, q_1][q_1, Z_0, q]$ (8)
$\Rightarrow baca[q_1, B, q_1][q_1, Z_0, q]$ (10)
$\Rightarrow bacab[q_1, Z_0, q]$ (11)
$\Rightarrow bacab\Lambda$ (12)

Production 12: $[q_1, Z_0, q_1] \rightarrow \Lambda$

Accepting and generating *bacab*

- The derivation:

$S \Rightarrow [q_0, Z_0, q_1]$ (0)
$\Rightarrow b[q_0, B, q_1][q_1, Z_0, q_1]$ (2)
$\Rightarrow ba[q_0, A, q_1][q_1, B, q_1][q_1, Z_0, q_1]$ (5)
$\Rightarrow bac[q_1, A, q_1][q_1, B, q_1][q_1, Z_0, q_1]$ (8)
$\Rightarrow baca[q_1, B, q_1][q_1, Z_0, q_1]$ (10)
$\Rightarrow bacab[q_1, Z_0, q_1]$ (11)
$\Rightarrow bacab$ (12)

Production 12: q gets bound to q_1

Accepting and generating *bacab*

- The derivation:

$S \Rightarrow [q_0, Z_0, q_1]$ (0)
$\Rightarrow b[q_0, B, q_1][q_1, Z_0, q_1]$ (2)
$\Rightarrow ba[q_0, A, q_1][q_1, B, q_1][q_1, Z_0, q_1]$ (5)
$\Rightarrow bac[q_1, A, q_1][q_1, B, q_1][q_1, Z_0, q_1]$ (8)
$\Rightarrow baca[q_1, B, q_1][q_1, Z_0, q_1]$ (10)
$\Rightarrow bacab[q_1, Z_0, q_1]$ (11)
$\Rightarrow bacab$ (12)

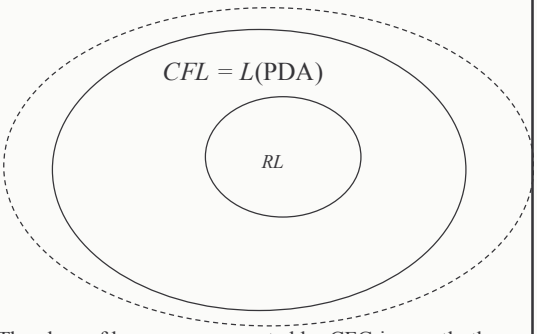
PDA and CFG

- ✓ There is a PDA M such that $L(M) = L(G)$ for every CFG G
- ✓ There is a CFG G such that $L(G) = L(M)$ for every PDA M
- The set of *CFL* generated by CFG (ambiguous or unambiguous) is the set of *CFL* accepted by PDA.

PDA and CFG

- ✓ There is a PDA M such that $L(M) = L(G)$ for every CFG G
- ✓ There is a CFG G such that $L(G) = L(M)$ for every PDA M
- ✓ The set of *CFL* generated by CFG (ambiguous or unambiguous) is the set of *CFL* accepted by PDA.

CFG ≡ PDA (Chomsky, 1961)



- The class of languages generated by CFG is exactly the class of languages accepted by PDA

Dr. Luis Pineda, IIMAS, UNAM & OSU-CIS, 2003