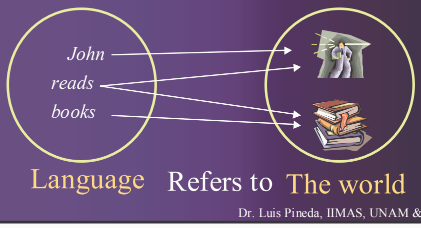
Session 3

Regular Languages and Expressions

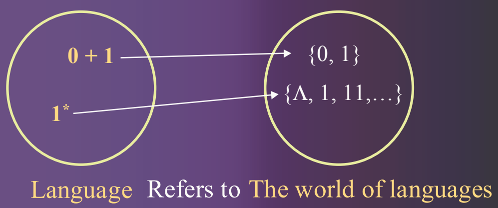
Syntax and Semantics

- Normally languages refer to individual objects, properties and relations in the world
- Linguistic symbols are *syntactic objects*
- The representation or reference relation:



Syntax and Semantics

- If language is the the object of study, we need a language to be able to talk about languages
- Sets of strings (languages) become *semantic objects*
- The representation or reference relation:



The language of Regular Expressions

- Syntax
  - Basic constants (for a given Σ)
    - Φ is regular expression (*RE*)
    - Λ is *RE*
    - If $a \in \Sigma$ then a is a *RE*
    - A variable, a italic capital letter (e.g. L), is a *RE*
  - Composition rules:
    - If E and F are *RE* then $E + F$ is *RE* (union)
    - If E and F are *RE* then EF is *RE* (concatenation)
    - If E is a *RE* then E^* is a *RE* (closure)
    - If E is a *RE* then (E) is a *RE* (introduction of parenthesis)
  - Only the expressions constructed by a FINITE application of the rules in this definition are *RE*

An alternative formulation

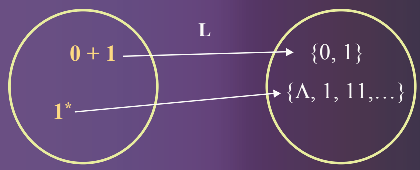
- Syntax
  - Basic constants (for a given Σ)
    - Φ is regular expression *RE*
    - Λ is *RE*
    - If $a \in \Sigma$ then a is a *RE*
    - A variable, a italic capital letter (e.g. L), is a *RE*
  - Composition rules (parenthesis are obligatory):
    - If E and F are *RE* then $(E + F)$ is *RE* (union)
    - If E and F are *RE* then (EF) is *RE* (concatenation)
    - If E is a *RE* then (E^*) is a *RE* (closure)
  - Only the expressions constructed by a FINITE application of the rules in this definition are *RE*

The language of Regular Expressions

- Semantics
  - Let *RE* the set of all regular expressions over Σ, R the set of regular languages and **L** an interpretation function from *RE* to R
  - Interpretation of basic constants
    - $\mathbf{L}(\Phi)$ is Φ (i.e. the empty language)
    - $\mathbf{L}(\Lambda)$ is $\{\Lambda\}$ (i.e. the language with the empty string)
    - If $a \in \Sigma$ then $\mathbf{L}(a)$ is $\{a\}$ {i.e. the language with a}
    - $\mathbf{L}(L)$ is any language
  - Interpretation of composite expressions:
    - $\mathbf{L}(E+F)$ is the union of $\mathbf{L}(E)$ and $\mathbf{L}(F)$
    - $\mathbf{L}(EF)$ or $\mathbf{L}(E.F)$ is the concatenation of $\mathbf{L}(E)$ and $\mathbf{L}(F)$
    - $\mathbf{L}(E^*)$ is $(\mathbf{L}(E))^*$ (i.e. the closure of $\mathrm{L}(E)$)
    - $\mathbf{L}((E))$ is $\mathbf{L}(E)$ (i.e. the same language)

Syntax and Semantics

- The representation or reference function **L**:



- The interpretation of a composite expression is a function of:
  - The interpretation of its constituent parts
  - The form of syntactic composition

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Examples

RE	Language
Λ:	{Λ}
0:	{0}
001:	{001}
0 + 1:	{0, 1}
0 + 10:	{0, 10}
(1 + Λ)001:	{1, Λ}{001}
(110)*(0 + 1):	$\{110\}^*\{0, 1\}$
1*10:	$\{1\}^*\{10\}$

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Examples

- … **(10 + 111 + 11010)***: $\{10, 111, 11010\}^*$
- **(0 + 10)*((11)* + 001 + Λ)**: $\{0, 10\}^*\{\{11\}^* \cup \{001, \Lambda\}\}$
- **01* + 1:** $\{0\}\{1\}^* \cup \{1\} = \{1, 0, 01, 011, \ldots, 011\ldots1\}$
- **(01)* + 1:** $\{01\}^* \cup \{1\} = \{1, \Lambda, 01, 0101, \ldots, 0101\ldots01\}$
- **0(1* + 1):** $\{0\}\{\{1\}^* \cup \{1\}\} = \{0, 01, 011, \ldots, 011\ldots1\}$

Note that: **0(1* + 1) = 01***

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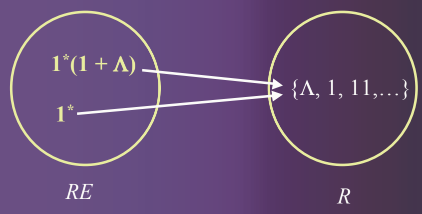
Equality of Regular Expression

- Regular expressions are equal if they refer to the same language:



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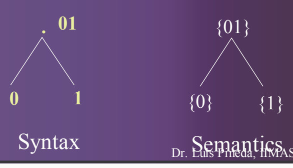
Example: The language of alternating 0's and 1's

- First: The language {01}: **01**



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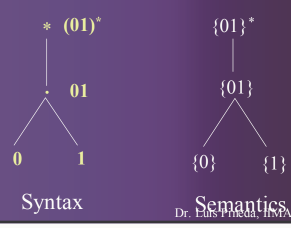
Example: The language of alternating 0's and 1's

- Second: The language $\{01\}^*$: **(01)***



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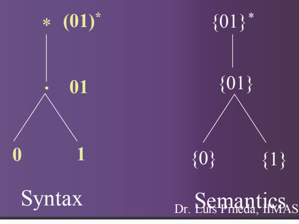
Example: The language of alternating 0's and 1's

- But we also need:
  - **{0101…0}**
  - **{1010…0}**
  - **{1010…1}**



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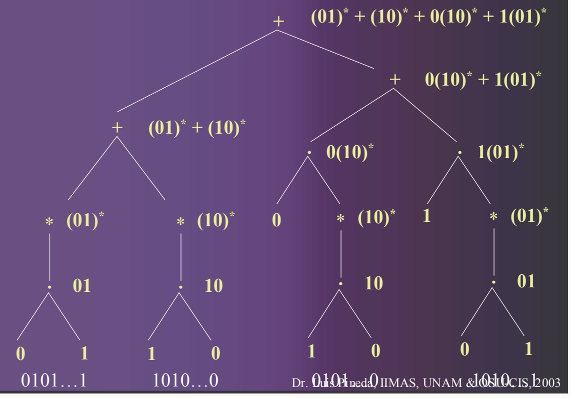
The language of alternating 0's and 1's



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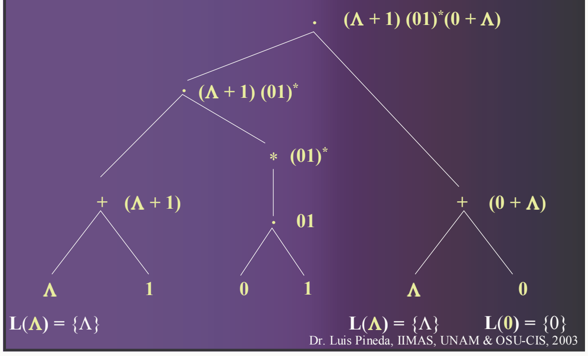
The expressive power of Λ



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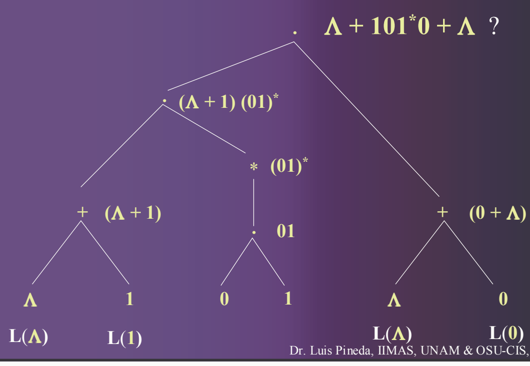
Are *RE* ambiguous?



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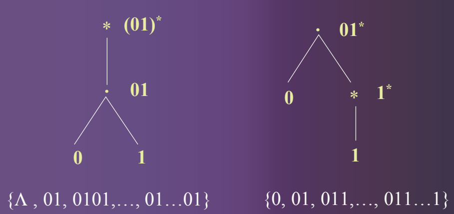
Structure of *RE*

- Operators apply to the structure below:



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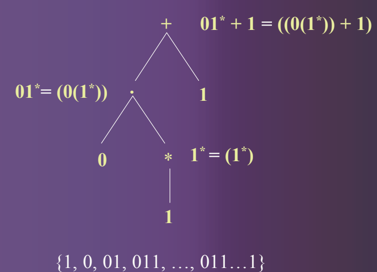
Structure and ambiguity

- "Ambiguity" of : **01* + 1**



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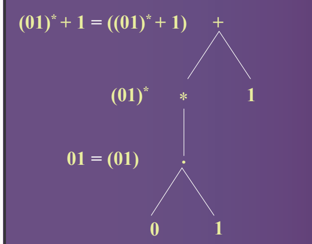
Structure and ambiguity

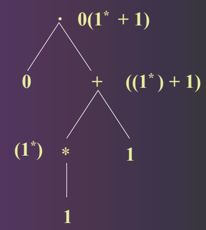
- “Ambiguity” of : $01^{*} + 1$



{1, Λ, 01, 0101,…, 0101…01}

Structure and ambiguity

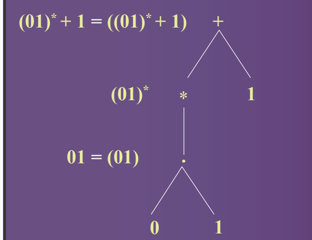
- “Ambiguity” of : $01^{*} + 1$

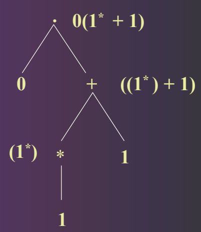


{0, 01, 011,…, 011…1}

Structure and ambiguity

- “Ambiguity” of : $01^{*} + 1$





{1, Λ, 01, 0101,…, 0101…01} {0, 01, 011,…, 011…1}

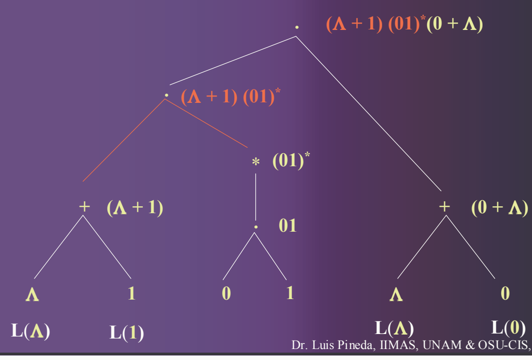
Precedence of operators

- Precedence order:
  - Highest: Star operator (*)
    - Applies to the smallest sequence to its left
  - Next: Concatenation operator (dot)
    - juxtaposition of strings
    - Strings with no other operator in between are grouped together
    - Associative (conventionally we group by the left)
  - Lowest: Union operator (+)
    - Associative (conventionally we group by the left)

Grouping “.” by the left

$(\Lambda + 1)\,(01)^{*}(0 + \Lambda)$

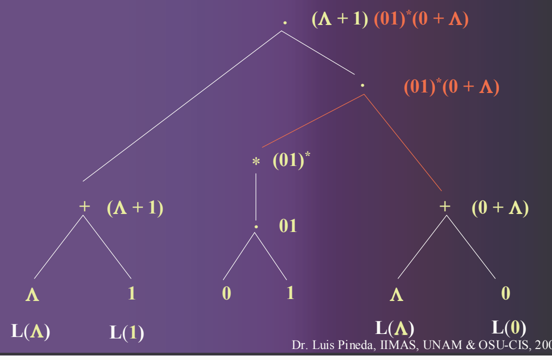
Grouping “.” by the left



Grouping "." by the right

$(\Lambda + 1)\ (01)^*(0 + \Lambda)$

Grouping "." by the right



Precedence, parenthesis and ambiguity

- *RE* can apparently be ambiguous, but
  - Parenthesis and precedence order eliminate ambiguity
- Also:
  - *RE* have a structure
  - Trees show the structure of *RE* explicitly!
  - An ambiguous expression have several possible structures
  - There is only one structure for every *RE*
- Looking at precedence rules and parenthesis, or at the structure of expressions, there is no ambiguity
- *RE* are NOT ambiguous!

Equality of Regular Expression

- Useful for simplifying expressions
- As we will see, useful for simplifying Automatas (with as few states as possible)
  - $1^*(1 + \Lambda) = 1^*$
  - $1^*1^* = 1^*$
  - $0^* + 1^* = 1^* + 0^*$
  - $(0^*1^*)^* = (0 + 1)^*$
  - $(0 + 1)^*01(0 + 1)^* + 1^*0^* = (0 + 1)^*$
- There is a general method (an algorithm) to decide whether two expressions define the same language

Simplifying *RE*

- $(r + s + rs + sr)^*$

Simplifying *RE*

- $(r + s + rs + sr)^*$
  - *rs* can be formed taking *r* and the *s*; similarly for *sr*, then:

    $(r + s + rs + sr)^* = (r + s)^*$

Simplifying *RE*

- $r(r^*r + r^*) + r^*$

Simplifying *RE*

- $r(r^*r + r^*) + r^*$
  - $r^*r = r^+$
  - $r(r^*r + r^*) + r^* = r(r^+ + r^*) + r^*$
    $= r(r^*) + r^*$
    $= rr^* + r^*$
    $= r^+ + r^*$
    $= r^*$

Interpreting *RE*

- Consider the two regular expressions:
  - $r = 0^* + 1^*$
  - $s = 01^* + 10^* + 1^*0 + (0^*1)^*$
- A string corresponding to r but not to s
  - 00
- A string corresponding to s but not to r
  - 01
- A string corresponding to both r and s
  - Several obvious ones: Λ, 0, 1
- A string in $\{0, 1\}^*$ corresponding to neither r or s
  - Any string of the form: 1^i0^i for $i \geq 2$

Finding *RE*

- Give a regular expression for the following language:
  - $\{ s \in \{a, b\}^* : |s|$ is not divisible by $2\}$
- If $|s|$ is divisible by 2
  - its length is even (i.e. otherwise its length is odd)
- *RE* for strings of even length:
  - $(aa + ab + ba + bb)^*$
- Adding one symbol, either a or b (i.e. odd length strings):
  - $(aa + ab + ba + bb)^*(a + b)$
- Alternatively:
  - $(a + b)(aa + ab + ba + bb)^*$
- Introducing abstraction:
  - $(a + b)((a + b)(a + b))^*$

Regular sets

- Languages built out:
  - Φ, Λ and all symbols in Σ
- By means of:
  - Union
  - Concatenation
  - Closure (Kleen-star)
- Through a *finite* number of operations!
  - A regular expression is itself finite string!
  - Formally, we don't allow ellipsis (…)
  - The tree of the expression is finite too!
- The resulting set, a language, is subset of the power set of Σ^* (2^{Σ^*}:), which cannot even be counted!
- There are many, many, sets that are not regular!

RE in the space of languages