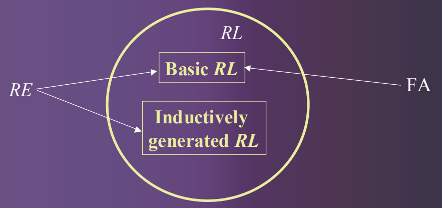
Session 7

Set Operations and FA

Descriptive power of *RE* and FA



We can define individual FA, so far!
But ee need to generate machines inductively!

Operations on FA

- According to Kleen's Theorem:
  - If L_1 and L_2 are both *RE* over Σ if and only if there are M_1 and M_2 accepting L_1 and L_2
- Definition of *RL*: IF L_1 and L_2 are *RL*
  - $L_1 \cup L_2$, L_1L_2 and L_1^* are also *RL*
- Set operations on *RL*
  - Union: $L_1 \cup L_2$ is a *RL*
  - Intersection: $L_1 \cap L_2$ is a *RL*
  - Difference: $L_1 - L_2$ is a *RL*
- Is there a way to generate machines for these languages from M_1 and M_2 (the machines accepting L_1 and L_2)?

Set operations on FA

- The case for $L_1 \cup L_2$
  - Consider whether a string $x \in L_1 \cup L_2$
  - It does if x belongs to either L_1 or L_2
  - Let M_1 and M_2

    $M_1 = (Q_1, \Sigma, q_1, A_1, \delta_1)$ such that $L(M_1) = L_1$

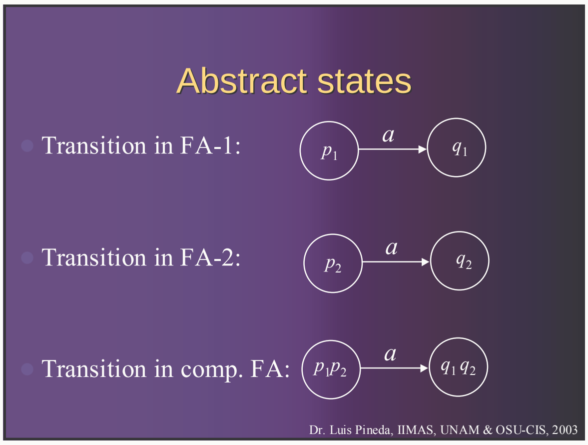
    $M_2 = (Q_2, \Sigma, q_2, A_2, \delta_2)$ such that $L(M_2) = L_2$
  - Process x with both M_1 and M_2
  - If x is accepted by either M_1 or M_2, it belongs to $L_1 \cup L_2$

Set operations on FA

- Process the string x **simultaneously** by M_1 and M_2
  - $\delta_1^*(q_1, x) = p$
  - $\delta_2^*(q_2, x) = q$
- The three machines:
  - If $p \in A_1$ OR $q \in A_2$ then x is accepted by the union of M_1 and M_2
  - If $p \in A_1$ AND $q \in A_2$ then x is accepted by the intersection of M_1 and M_2
  - If $p \in A_1$ AND $q \notin A_2$ then x is accepted by the difference M_1 but not M_2

Set operations on FA

- The process step by step:
  - $\delta_1(p, a) = r$
  - $\delta_2(q, a) = s$
- The composite machine can be though in terms of abstract pairs of states (p, q) on the symbol a:
  - It moves from state (p, q) to $(\delta_1(p, a), \delta_2(q, a))$



Abstract states

- Transition in FA-1:
- Transition in FA-2:
- Transition in comp. FA:

Theorem (constructive)

- Suppose
  - $M_1 = (Q_1, \Sigma, q_1, A_1, \delta_1)$ such that $\mathbf{L}(M_1) = L_1$
  - $M_2 = (Q_2, \Sigma, q_2, A_2, \delta_2)$ such that $\mathbf{L}(M_2) = L_2$
- Let $M = (Q, \Sigma, q_0, A, \delta)$ be a construction from M_1 and M_2 as follows:
  - $Q = Q_1 \times Q_2$
  - $q_0 = (q_1, q_2)$
  - For any $p \in Q_1$, $q \in Q_2$ and $a \in \Sigma$:

$$\delta((p, q), a) = (\delta_1(p, a), \delta_2(q, a))$$

- Then
  - If $A = \{(p, q) \mid p \in A_1 \text{ OR } q \in A_2\}$ then M accepts $L_1 \cup L_2$
  - If $A = \{(p, q) \mid p \in A_1 \text{ AND } q \in A_2\}$ then M accepts $L_1 \cap L_2$
  - If $A = \{(p, q) \mid p \in A_1 \text{ AND } q \notin A_2\}$ then M accepts $L_1 - L_2$

Proof

- Acceptance of M_1 and M_2 is defined in terms of $\delta_1{}^*$ and $\delta_2{}^*$, and acceptance by M is defined in terms of δ^*:
  - For $x \in \Sigma^*$ and $(p, q) \in Q$:

$$\delta^*((p, q), x) = (\delta_1{}^*(p, x), \delta_2{}^*(q, x))$$

- A string x is accepted by M if and only if

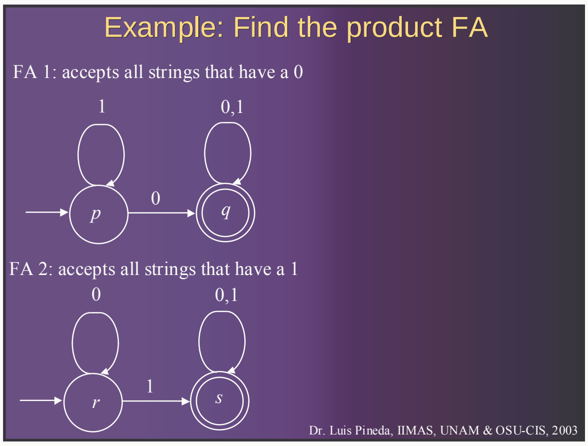
$$\delta^*((q_1, q_2), x) \in A$$

(i.e. Takes M from the initial to an accepting state)

- This is true if and only if

$$(\delta_1{}^*(q_1, x), \delta_2{}^*(q_2, x)) \in A$$

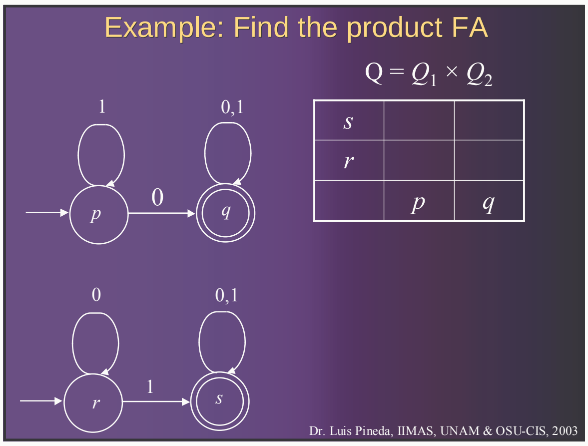
- The set A is defined for case 1 as saying that
  - $\delta_1{}^*(q_1, x) \in \boldsymbol{A}_1$ OR $\delta_2{}^*(q_2, x) \in A_2$
  - Similarly for the other two cases



Example: Find the product FA

FA 1: accepts all strings that have a 0

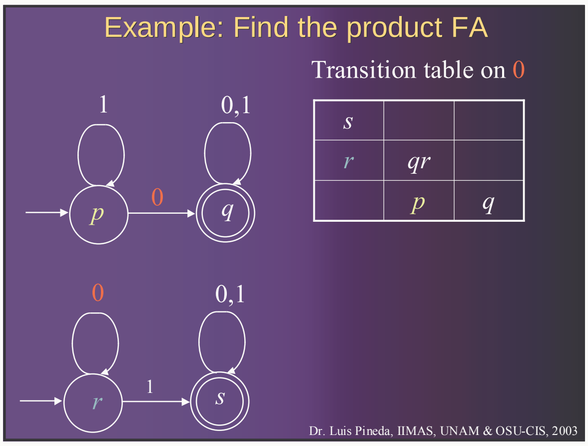
FA 2: accepts all strings that have a 1



Example: Find the product FA

$Q = Q_1 \times Q_2$

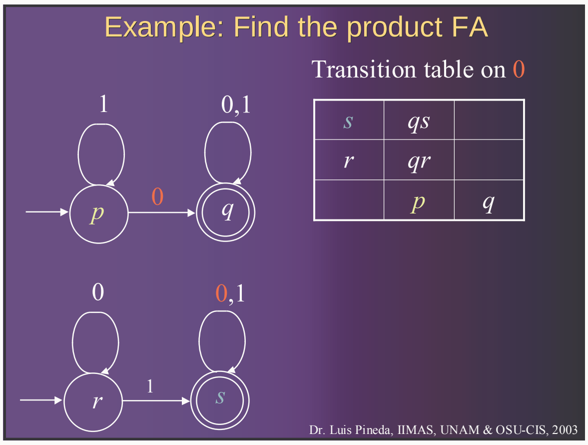
s		
r		
	p	q



Example: Find the product FA

Transition table on 0

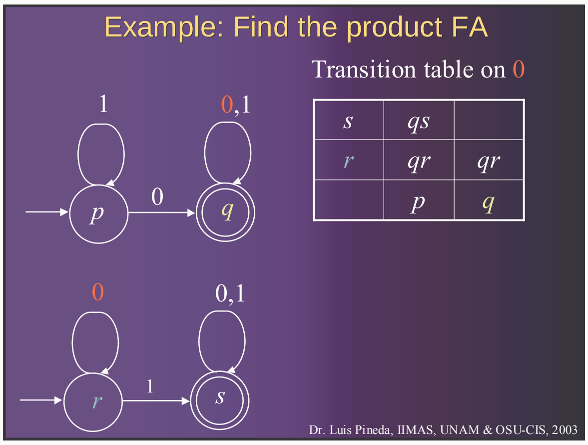
s		
r	qr	
	p	q



Example: Find the product FA

Transition table on 0

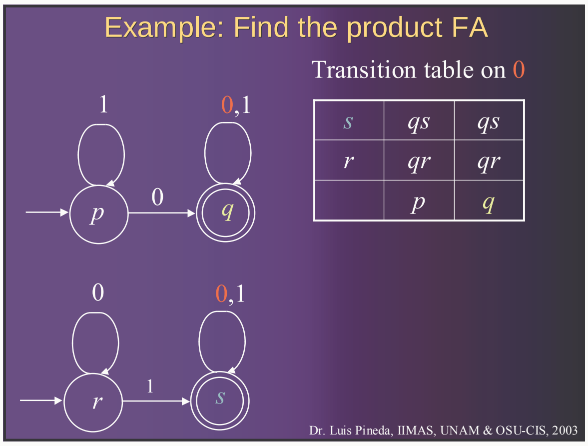
s	*qs*	
r	*qr*	
	p	*q*



Example: Find the product FA

Transition table on 0

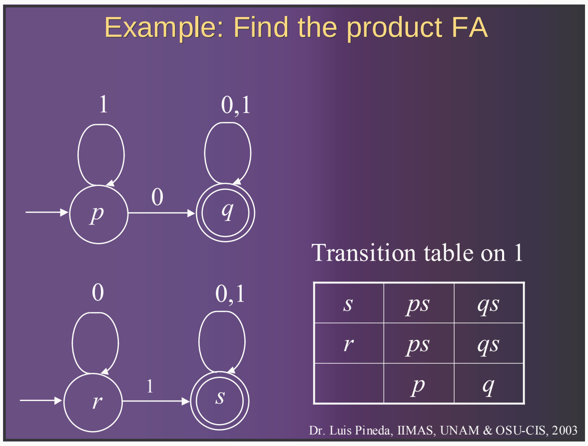
s	*qs*	
r	*qr*	*qr*
	p	*q*



Example: Find the product FA

Transition table on 0

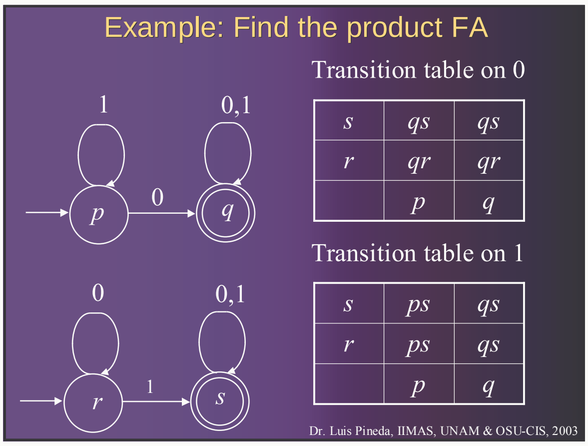
s	*qs*	*qs*
r	*qr*	*qr*
	p	*q*



Example: Find the product FA

Transition table on 1

s	*ps*	*qs*
r	*ps*	*qs*
	p	*q*



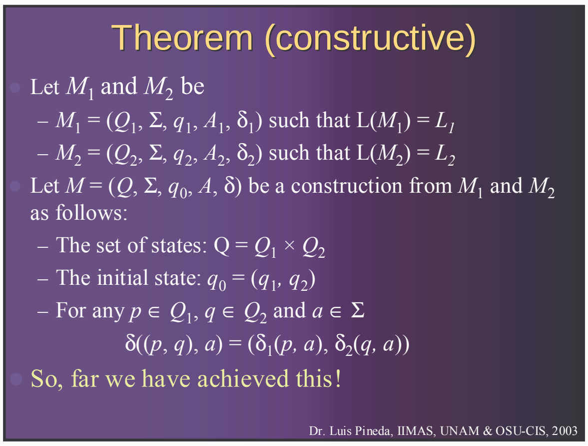
Example: Find the product FA

Transition table on 0

s	*qs*	*qs*
r	*qr*	*qr*
	p	*q*

Transition table on 1

s	*ps*	*qs*
r	*ps*	*qs*
	p	*q*

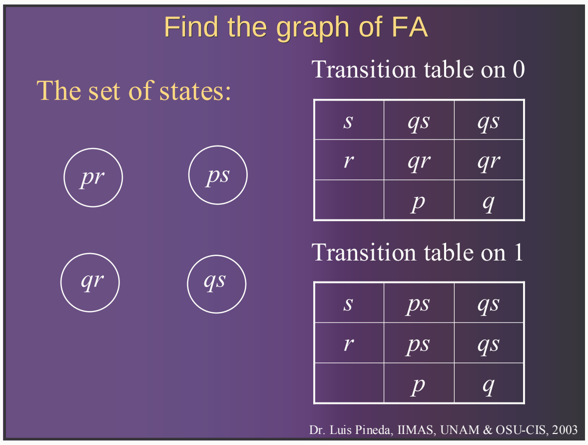


Theorem (constructive)

- Let M_1 and M_2 be
  - $M_1 = (Q_1, \Sigma, q_1, A_1, \delta_1)$ such that $L(M_1) = L_1$
  - $M_2 = (Q_2, \Sigma, q_2, A_2, \delta_2)$ such that $L(M_2) = L_2$
- Let $M = (Q, \Sigma, q_0, A, \delta)$ be a construction from M_1 and M_2 as follows:
  - The set of states: $Q = Q_1 \times Q_2$
  - The initial state: $q_0 = (q_1, q_2)$
  - For any $p \in Q_1$, $q \in Q_2$ and $a \in \Sigma$

$$\delta((p, q), a) = (\delta_1(p, a), \delta_2(q, a))$$

- So, far we have achieved this!



Find the graph of FA

The set of states:

Transition table on 0

s	*qs*	*qs*
r	*qr*	*qr*
	p	*q*

Transition table on 1

s	*ps*	*qs*
r	*ps*	*qs*
	p	*q*

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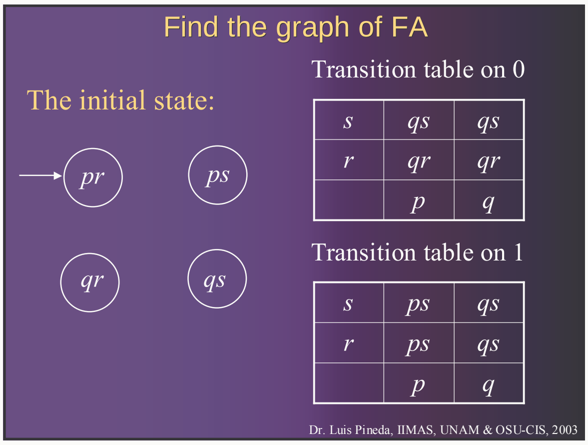

Find the graph of FA

The initial state:

Transition table on 0

s	*qs*	*qs*
r	*qr*	*qr*
	p	*q*

Transition table on 1

s	*ps*	*qs*
r	*ps*	*qs*
	p	*q*

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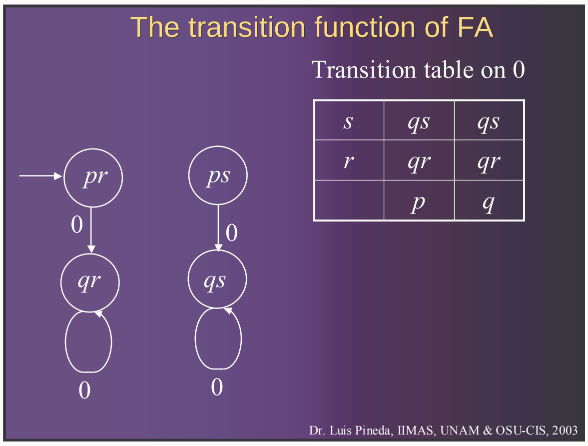

The transition function of FA

Transition table on 0

s	*qs*	*qs*
r	*qr*	*qr*
	p	*q*

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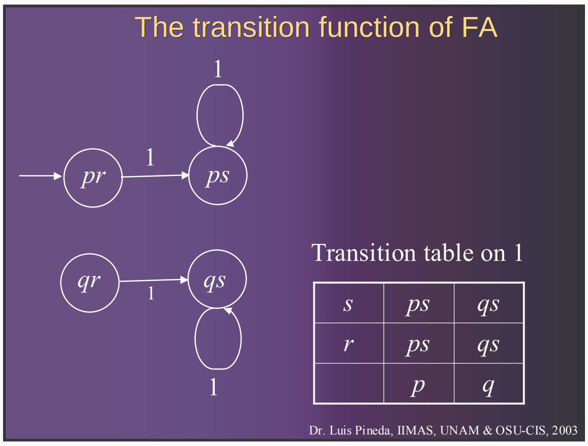

The transition function of FA

Transition table on 1

s	*ps*	*qs*
r	*ps*	*qs*
	p	*q*

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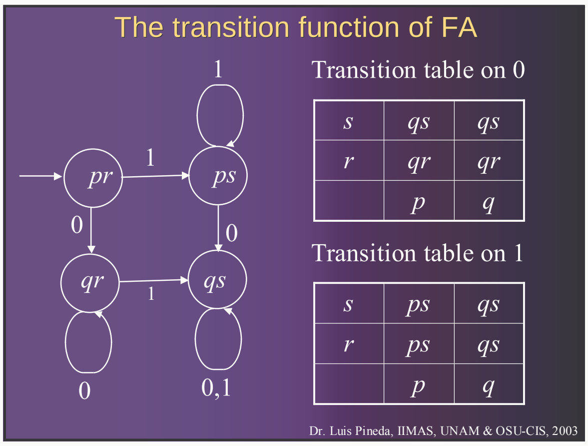

The transition function of FA

Transition table on 0

s	*qs*	*qs*
r	*qr*	*qr*
	p	*q*

Transition table on 1

s	*ps*	*qs*
r	*ps*	*qs*
	p	*q*

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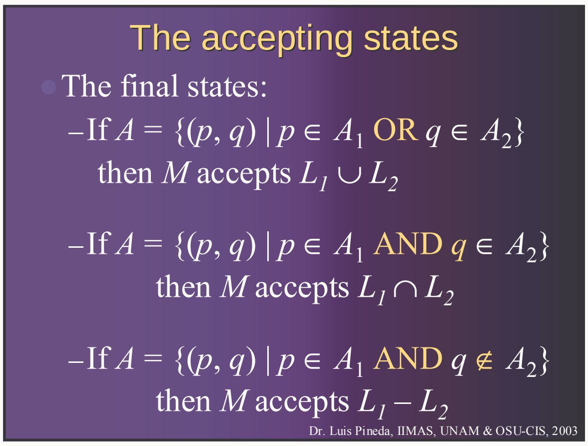

The accepting states

- The final states:
  - If $A = \{(p, q) \mid p \in A_1 \text{ OR } q \in A_2\}$ then M accepts $L_1 \cup L_2$
  - If $A = \{(p, q) \mid p \in A_1 \text{ AND } q \in A_2\}$ then M accepts $L_1 \cap L_2$
  - If $A = \{(p, q) \mid p \in A_1 \text{ AND } q \notin A_2\}$ then M accepts $L_1 - L_2$

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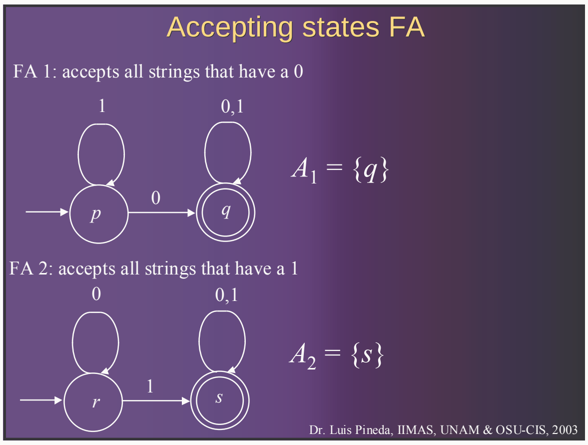


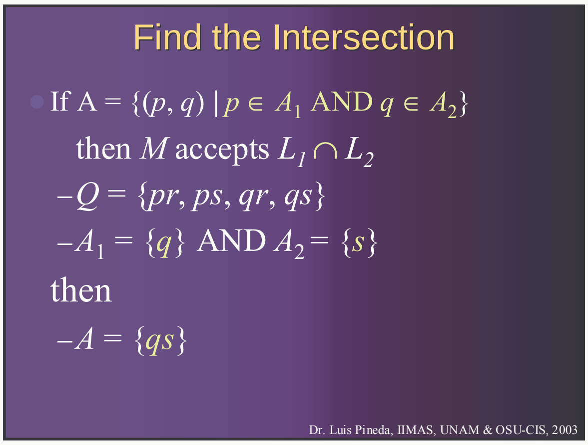
Accepting states FA

FA 1: accepts all strings that have a 0

$A_1 = \{q\}$

FA 2: accepts all strings that have a 1

$A_2 = \{s\}$



Find the Intersection

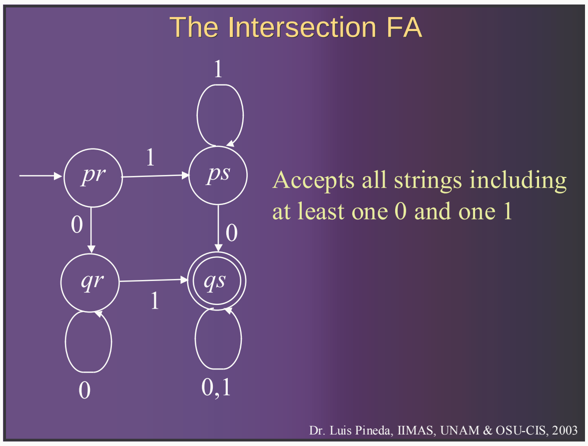
- If A = $\{(p, q) \mid p \in A_1$ AND $q \in A_2\}$

  then M accepts $L_1 \cap L_2$

  - $Q = \{pr, ps, qr, qs\}$
  - $A_1 = \{q\}$ AND $A_2 = \{s\}$

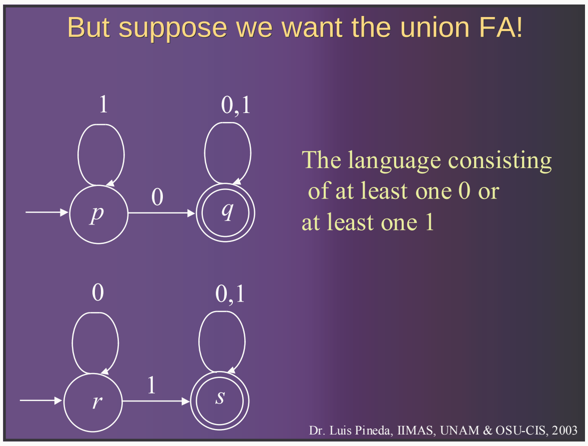
  then

  - $A = \{qs\}$



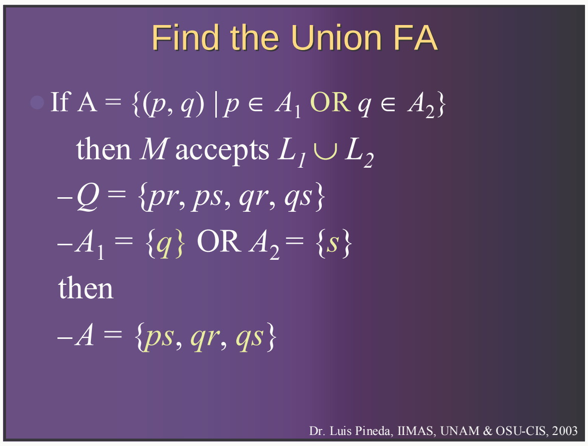
The Intersection FA

Accepts all strings including at least one 0 and one 1



But suppose we want the union FA!

The language consisting of at least one 0 or at least one 1



Find the Union FA

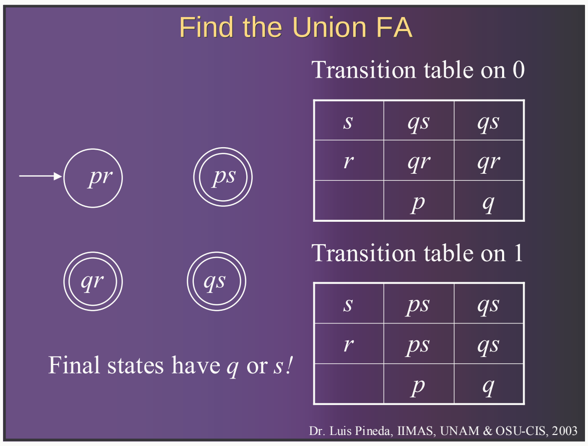
- If A = $\{(p, q) \mid p \in A_1$ OR $q \in A_2\}$

  then M accepts $L_1 \cup L_2$

  - $Q = \{pr, ps, qr, qs\}$
  - $A_1 = \{q\}$ OR $A_2 = \{s\}$

  then

  - $A = \{ps, qr, qs\}$



Find the Union FA

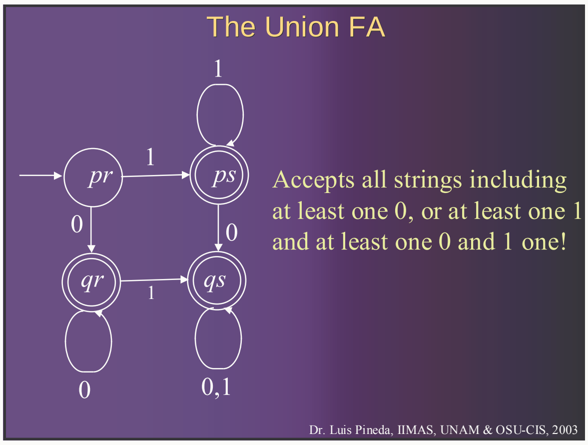
Transition table on 0

s	*qs*	*qs*
r	*qr*	*qr*
	p	*q*

Transition table on 1

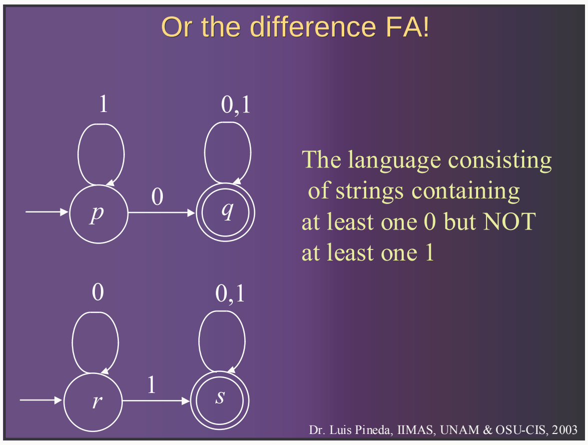
s	*ps*	*qs*
r	*ps*	*qs*
	p	*q*

Final states have *q* or *s*!



The Union FA

Accepts all strings including at least one 0, or at least one 1 and at least one 0 and 1 one!



Or the difference FA!

The language consisting of strings containing at least one 0 but NOT at least one 1

Find the Difference FA

- If $A = \{(p, q) \mid p \in A_1$ AND $q \notin A_2\}$

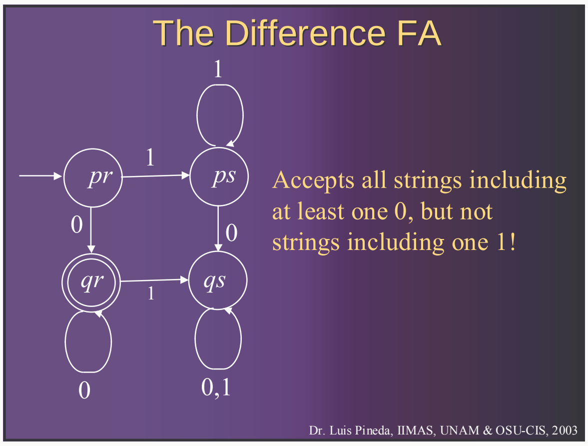
  then M accepts $L_1 - L_2$

  – $Q = \{pr, ps, qr, qs\}$

  – $A_1 = \{q\}$ BUT NOT $A_2 = \{s\}$

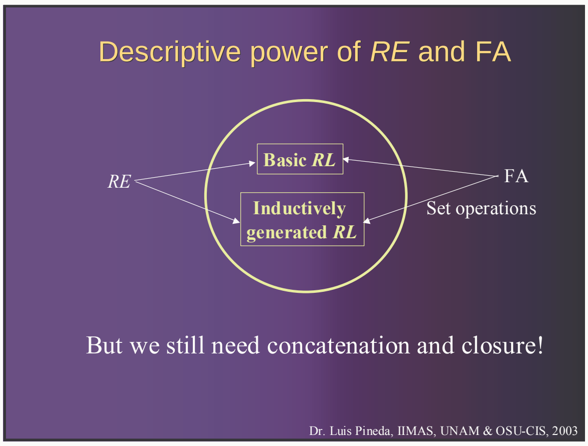
  then

  – $A = \{qr\}$



The Difference FA

Accepts all strings including at least one 0, but not strings including one 1!



Descriptive power of *RE* and FA

RE

Basic *RL*

Inductively generated *RL*

FA

Set operations

But we still need concatenation and closure!