Session 9

NFA are FA: The subset construction

NFA are FA: the subset construction

- For any NFA $M = (Q, \Sigma, q_0, A, \delta)$ accepting the language $L \subseteq \Sigma^*$, there is a FA $M_1 = (Q_1, \Sigma, q_1, A_1, \delta_1)$ that accepts L
- Search algorithms illustrate how all branches of a non-deterministic path can be explore deterministically
  - M: NFA is an specification
  - M_1: DFA is the implementation!
  - Given an specification it is always possible to find an implementation (an algorithm) automatically!
- For all NFA there is an algorithm to find its equivalent DFA

NFA are FA: the subset construction

- Proof (constructive): Find a DFA M_1 from the definition of an NFA M:
- M_1 is defined as follows:
  - The set of states of M_1: $Q_1 = 2^Q$,
  - The initial state of M_1: $q_1 = \{q_0\}$
  - The transition function of M_1:

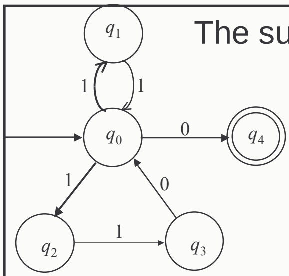
    For $q \in Q_1$ and $a \in \Sigma$

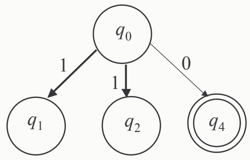
$$\delta_1(q, a) = \bigcup_{r \in q} \delta(r, a)$$

  - The set of accepting states of M_1:

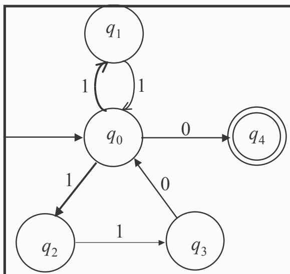
    $A_1 = \{q \in Q_1 \mid q \cap A \neq \emptyset\}$

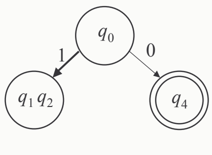
The subset construction:





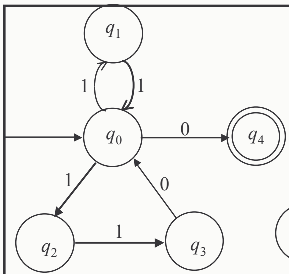
$\delta(q_0, 1) = \{q_1, q_2\}$
$\delta(q_0, 0) = \{q_4\}$

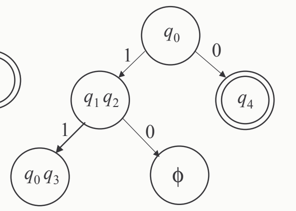




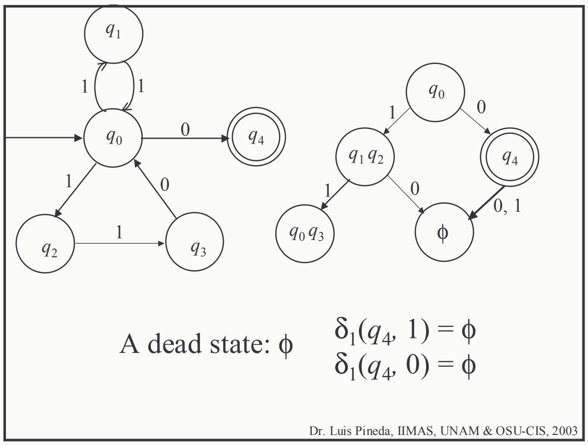
$\delta_1(q_0, 1) = q_1q_2$
$\delta_1(q_0, 0) = q_4$

- Collapsing the states:
  - Making an abstract state
  - A deterministic transition!





$\delta_1(q_1q_2, 1) = q_0q_3$
$\delta_1(q_1q_2, 0) = \phi$


A dead state: ϕ
δ1(q4, 1) = ϕ
δ1(q4, 0) = ϕ
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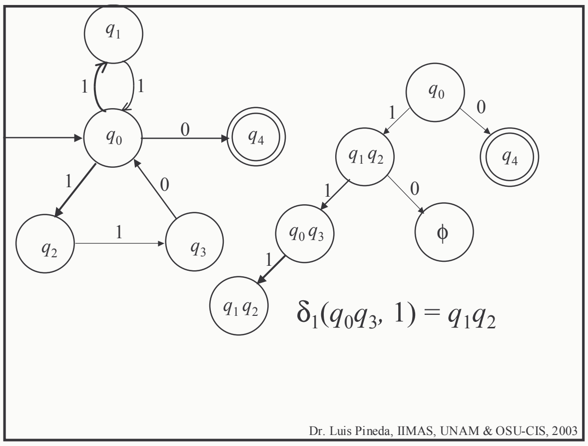
δ1(q0q3, 1) = q1q2
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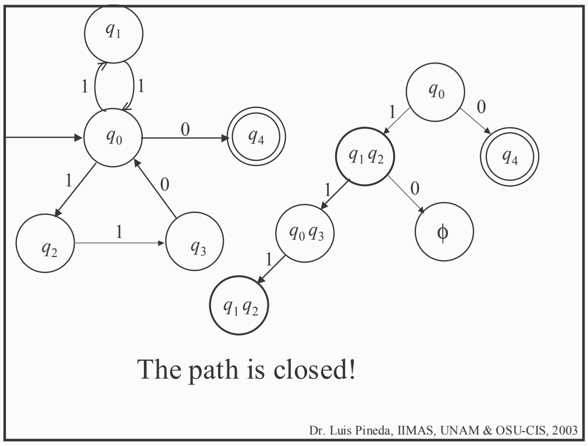
The path is closed!
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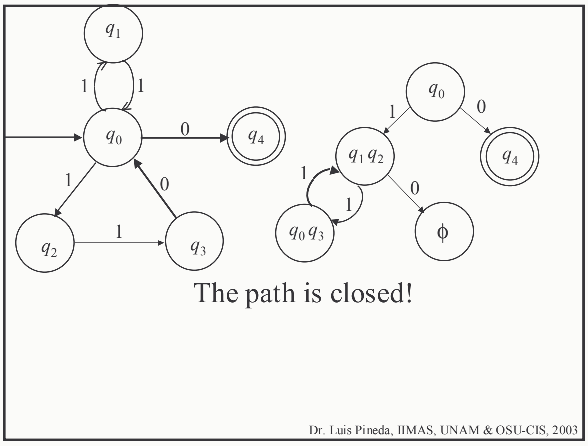
The path is closed!
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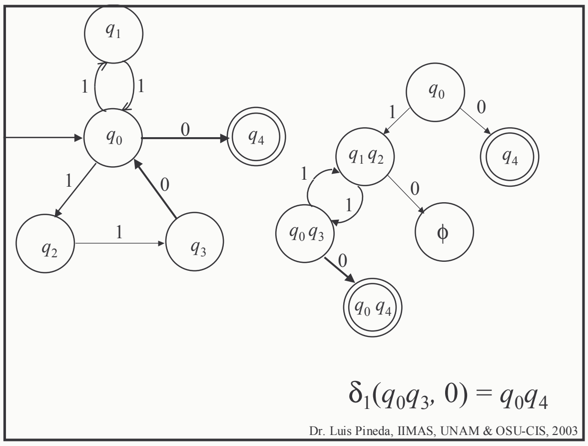
δ1(q0q3, 0) = q0q4
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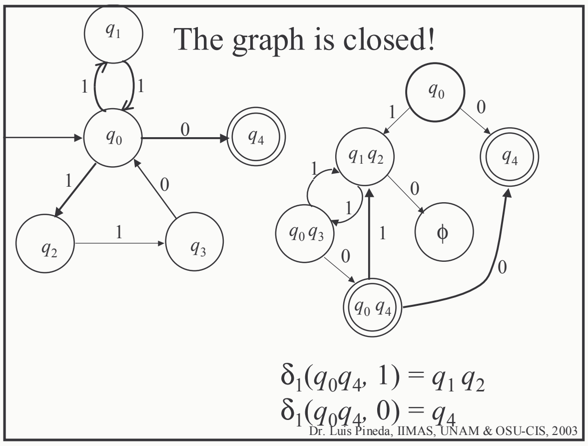
The graph is closed!
δ1(q0q4, 1) = q1 q2
δ1(q0q4, 0) = q4
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The states of the subset construction

- The the states (and names) of M_1 are defined out of the states (and their names) of M:
- M_1 is defined as follows:
  - The set of states of M_1: $Q_1 = 2^Q$,
  - The initial state of M_1: $q_1 = \{q_0\}$
- If M has 5 states:
  - Then, M_1 has $2^5 = 32$ possible states!
  - Fortunately we only needed 6!

	0	1
$\rightarrow q_0$	$\{q_4\}$	$\{q_1, q_2\}$
q_1	ϕ	$\{q_0\}$
q_2	ϕ	$\{q_3\}$
q_3	$\{q_0\}$	ϕ
$*q_4$	ϕ	ϕ

The power set 2^Q

- No. the sets of n objects taken r at the time

$$C_r^n = \frac{n!}{r!(n-r)!}$$

- M has 5 states so, we have:
  - The empty state: ϕ
  - 5 states made of 1 state: $\{q_0\}, \{q_1\}, \{q_2\}, \{q_3\}, \{q_4\}$
  - 10 states made of 2 states: $\{q_0 q_1\}, \{q_0 q_2\}, \{q_0 q_3\}, \ldots$
  - 10 states made of 3 states: $\{q_0 q_1 q_2\}, \{q_0 q_2 q_3\}, \{q_0 q_3 q_4\}, \ldots$
  - 5 states made of 4 states: $\{q_0 q_1 q_2 q_3\}, \{q_0 q_1 q_2 q_4\}, \ldots$
  - 1 state made of 5 states: $\{q_0 q_1 q_2 q_3 q_5\}$

A binary code for the names of states in M_1

- The state composed of q_1 and q_2:

States of M	q_0	q_1	q_2	q_3	q_4
Name of state in M_1	0	1	0	1	0

- There $2^{|Q|}$ possible states in M_1 (32 in this case)
- Each binary numeral from 0 to 31 corresponds to the name of one of these states

The possible entries in δ_1:

$q_0q_1q_2q_3q_4$	0	1
00000		
00001		
00010		
00011		
00100		
00101		
00110		
00111		

$q_0q_1q_2q_3q_4$	0	1
01000		
01001		
01010		
01011		
01100		
01101		
01110		
01111		

ie. 01011 is the state $0q_10q_3q_4 = q_1q_3q_4$

The possible entries in δ_1:

$q_0q_1q_2q_3q_4$	0	1
10000		
10001		
10010		
10011		
10100		
10101		
10111		
10111		

$q_0q_1q_2q_3q_4$	0	1
11000		
11001		
11010		
11011		
11100		
11101		
11110		
11111		

NFA are FA: the subset construction

- The set of states of M_1: $Q_1 = 2^Q$,
- The initial state of M_1: $q_1 = \{q_0\}$
  - $q_1 = \{q_0\} = 10000$
- The set of accepting states of M_1:

  $A_1 = \{q \in Q_1 \mid q \cap A \neq \emptyset\}$

  $A = \{q_4\}$
  - All states whose name has the pattern XXXX1 have a q_4, and are accepting states

The NFA M:

	0	1
$\rightarrow q_0$	$\{q_4\}$	$\{q_1, q_2\}$
q_1	ϕ	$\{q_0\}$
q_2	ϕ	$\{q_3\}$
q_3	$\{q_0\}$	ϕ
$*q_4$	ϕ	ϕ

NFA are FA: the subset construction

- The transition function of M_1:

For $q \in Q_1$ and $a \in \Sigma$ $\quad \delta_1(q,a) = \bigcup_{r \in q} \delta(r,a)$

	0	1
$\rightarrow q_0$	$\{q_4\}$	$\{q_1, q_2\}$
q_1	ϕ	$\{q_0\}$
q_2	ϕ	$\{q_3\}$
q_3	$\{q_0\}$	ϕ
$*q_4$	ϕ	ϕ

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Definition of δ_1

$q_0q_1q_2q_3q_4$	0	1
00000		
$\rightarrow$10000		

	0	1
$\rightarrow q_0$	$\{q_4\}$	$\{q_1, q_2\}$
q_1	ϕ	$\{q_0\}$
q_2	ϕ	$\{q_3\}$
q_3	$\{q_0\}$	ϕ
$*q_4$	ϕ	ϕ

00000 = ϕ, q_0 = 10000, q_1 = 01000 and q_4 = 00001

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Definition of δ_1

$q_0q_1q_2q_3q_4$	0	1
00000		
$\rightarrow$10000	00001	01100

	0	1
$\rightarrow q_0$	$\{q_4\}$	$\{q_1, q_2\}$
q_1	ϕ	$\{q_0\}$
q_2	ϕ	$\{q_3\}$
q_3	$\{q_0\}$	ϕ
$*q_4$	ϕ	ϕ

$\delta_1(10000, 0) = \delta(q_0, 0) = \{q_4\}$

$\delta_1(10000, 1) = \delta(q_0, 1) = \{q_1, q_2\}$

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Definition of δ_1

$q_0q_1q_2q_3q_4$	0	1
00000		
$\rightarrow$10000	00001	01100
*00001	ϕ	ϕ

	0	1
$\rightarrow q_0$	$\{q_4\}$	$\{q_1, q_2\}$
q_1	ϕ	$\{q_0\}$
q_2	ϕ	$\{q_3\}$
q_3	$\{q_0\}$	ϕ
$*q_4$	ϕ	ϕ

$\delta_1(00001, 0) = \delta(q_4, 0) = \phi$

$\delta_1(00001, 1) = \delta(q_1, 1) = \phi$

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Definition of δ_1

$q_0q_1q_2q_3q_4$	0	1
00000	ϕ	ϕ
$\rightarrow$10000	00001	01100
*00001	ϕ	ϕ

	0	1
$\rightarrow q_0$	$\{q_4\}$	$\{q_1, q_2\}$
q_1	ϕ	$\{q_0\}$
q_2	ϕ	$\{q_3\}$
q_3	$\{q_0\}$	ϕ
$*q_4$	ϕ	ϕ

$\delta_1(00000, 0) = \phi$

$\delta_1(00000, 1) = \phi$

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Definition of δ_1

$q_0q_1q_2q_3q_4$	0	1
00000	ϕ	ϕ
$\rightarrow$10000	00001	01100
*00001	ϕ	ϕ
01100	ϕ	10010

	0	1
$\rightarrow q_0$	$\{q_4\}$	$\{q_1, q_2\}$
q_1	ϕ	$\{q_0\}$
q_2	ϕ	$\{q_3\}$
q_3	$\{q_0\}$	ϕ
$*q_4$	ϕ	ϕ

$\delta_1(01100, 0) = \delta(q_1, 0) \cup \delta(q_2, 0) = \phi$

$\delta_1(01100, 1) = \delta(q_1, 1) \cup \delta(q_2, 1) = \{q_0\} \cup \{q_3\} = \{q_0, q_3\}$

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Definition of δ_1

$q_0q_1q_2q_3q_4$	0	1
00000	ϕ	ϕ
→10000	00001	01100
*00001	ϕ	ϕ
01100	ϕ	10010
10010	10001	01100

	0	1
$\to q_0$	$\{q_4\}$	$\{q_1, q_2\}$
q_1	ϕ	$\{q_0\}$
q_2	ϕ	$\{q_3\}$
q_3	$\{q_0\}$	ϕ
$*q_4$	ϕ	ϕ

$\delta_1(10010, 0) = \delta(q_0, 0) \cup \delta(q_3, 0) = \{q_4\} \cup \{q_0\} = \{q_4, q_0\}$

$\delta_1(10010, 1) = \delta(q_0, 1) \cup \delta(q_3, 1) = \{q_1, q_2\}$

Definition of δ_1

$q_0q_1q_2q_3q_4$	0	1
00000	ϕ	ϕ
→10000	00001	01100
*00001	ϕ	ϕ
01100	ϕ	10010
10010	10001	01100
*10001	00001	01100

	0	1
$\to q_0$	$\{q_4\}$	$\{q_1, q_2\}$
q_1	ϕ	$\{q_0\}$
q_2	ϕ	$\{q_3\}$
q_3	$\{q_0\}$	ϕ
$*q_4$	ϕ	ϕ

$\delta_1(10001, 0) = \delta(q_0, 0) \cup \delta(q_4, 0) = \{q_4\}$

$\delta_1(10001, 1) = \delta(q_0, 1) \cup \delta(q_4, 1) = \{q_1, q_2\}$

Definition of δ_1

$q_0q_1q_2q_3q_4$	0	1
00000	ϕ	ϕ
→10000	00001	01100
*00001	ϕ	ϕ
01100	ϕ	10010
10010	10001	01100
*10001	00001	01100

	0	1
$\to q_0$	$\{q_4\}$	$\{q_1, q_2\}$
q_1	ϕ	$\{q_0\}$
q_2	ϕ	$\{q_3\}$
q_3	$\{q_0\}$	ϕ
$*q_4$	ϕ	ϕ

We get closure: There are no more possible transitions!

The resulting DFA

$q_0q_1q_2q_3q_4$	Names FA	0	1
00000	s	ϕ	ϕ
→10000	$\to q_0$	00001	01100
*00001	$*p$	ϕ	ϕ
01100	r	ϕ	10010
10010	t	10001	01100
*10001	$*u$	00001	01100

Renaming the states

Renaming the states!

$q_0q_1q_2q_3q_4$	Names FA	0	1
00000	s	s = ϕ	s = ϕ
→10000	$\to q_0$	p = 00001	r = 01100
*00001	$*p$	s = ϕ	s = ϕ
01100	r	s = ϕ	t = 10010
10010	t	u = 10001	r = 01100
*10001	$*u$	p = 00001	r = 01100

Renaming the states!

$q_0q_1q_2q_3q_4$	Names FA	0	1
00000	s	s	s
→10000	$\to q_0$	p	r
*00001	$*p$	s	s
01100	r	s	t
10010	t	u	r
*10001	$*u$	p	r

Renaming the states!

	0	1
s	s	s
$\rightarrow q_0$	p	r
$*p$	s	s
r	s	t
t	u	r
$*u$	p	r

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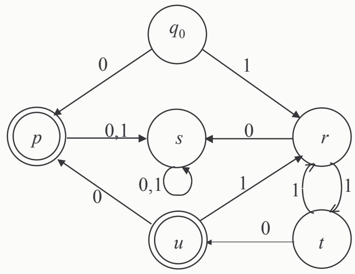
The equivalent DFA!

	0	1
s	s	s
$\rightarrow q_0$	p	r
$*p$	s	s
r	s	t
t	u	r
$*u$	p	r



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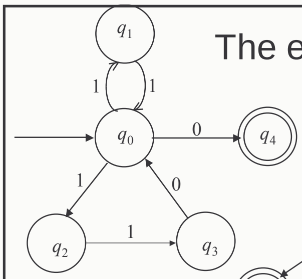
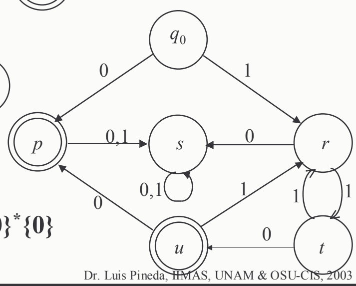
The equivalent DFA!





Accepting: $\{\mathbf{11,110}\}^*\{\mathbf{0}\}$

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NFA are FA

- M_1 accepts the same language as M follows from the fact that for any $x \in \Sigma^*$

$$\delta_1^*(q_1, x) = \delta^*(q_0, x)$$

- This is proved by induction!

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NFA are FA: the subset construction

- For any NFA $M = (Q, \Sigma, q_0, A, \delta)$ accepting a language $L \subseteq \Sigma^*$, there is a FA $M_1 = (Q_1, \Sigma, q_1, A_1, \delta_1)$ that accepts L
- M_1 is defined as follows:
  - The set of states of M_1: $Q_1 = 2^Q$,
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  - The transition function of M_1:

    For $q \in Q_1$ and $a \in \Sigma$

$$\delta_1(q, a) = \bigcup_{r \in q} \delta(r, a)$$

  - The set of accepting states of M_1:

    $A_1 = \{q \in Q_1 \mid q \cap A \neq \emptyset\}$

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$$\delta_1^*(q_1, x) = \delta^*(q_0, x)$$

- The base case: If $x = \Lambda$

$$\delta_1^*(q_1, x) = \delta_1^*(q_1, \Lambda)$$

$= q_1$ (by definition of δ_1^*)

$= \{q_0\}$ (by definition of q_1)

$= \delta^*(q_0, \Lambda)$ (by. def. of δ^*)

$= \delta^*(q_0, x)$

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- The induction hypothesis:

$$\delta_1^*(q_1, x) = \delta^*(q_0, x)$$

- We wish to prove that for any $a \in \Sigma$,

$$\delta_1^*(q_1, xa) = \delta^*(q_0, xa)$$

- $\delta_1^*(q_1, xa) = \delta_1(\delta_1^*(q_1, x), a)$ (by definition of δ_1^*)

$= \delta_1(\delta^*(q_0, x), a)$ (by the induction hypothesis)

$= \bigcup_{r \in \delta^*(q_0, x)} \delta(r, a)$ (by def. of δ_1: the subset construction)

$= \delta^*(q_0, xa)$ (by def. of δ^*)

M and M_1 recognize the same language

- A string x is accepted by M_1 if $\delta_1^*(q_1, x) \in A_1$
- this is true if and only if $\delta^*(q_0, x) \in A$
- By definition of A this is true if and only if

$$\delta^*(q_0, x) \cap A \neq \Phi$$

- So, x is accepted by M_1 if and only if x is accepted by M

The third description of *RL*