Conservation principles and action schemes in the synthesis of geometric concepts

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Diagrammatic reasoning

- Reasoning
- Learning
- Perception
- Design and creativity
- Theorem proving
- Ubiquitous in science and engineering

Diagrammatic reasoning

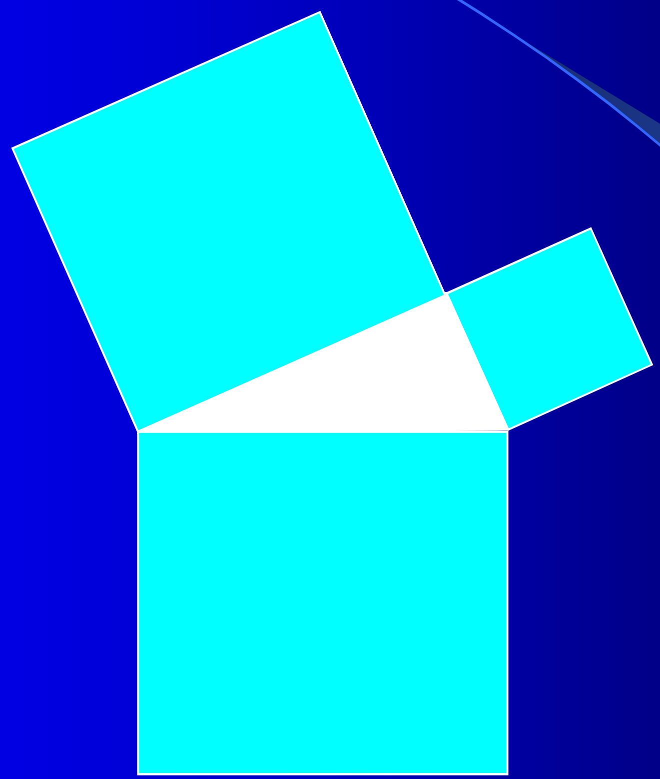
- How diagrammatic knowledge is represented
- What kind of inferences are supported by diagrams
- How external representations participate in this process

This is a problem in knowledge representation!

Some general questions about diagrams

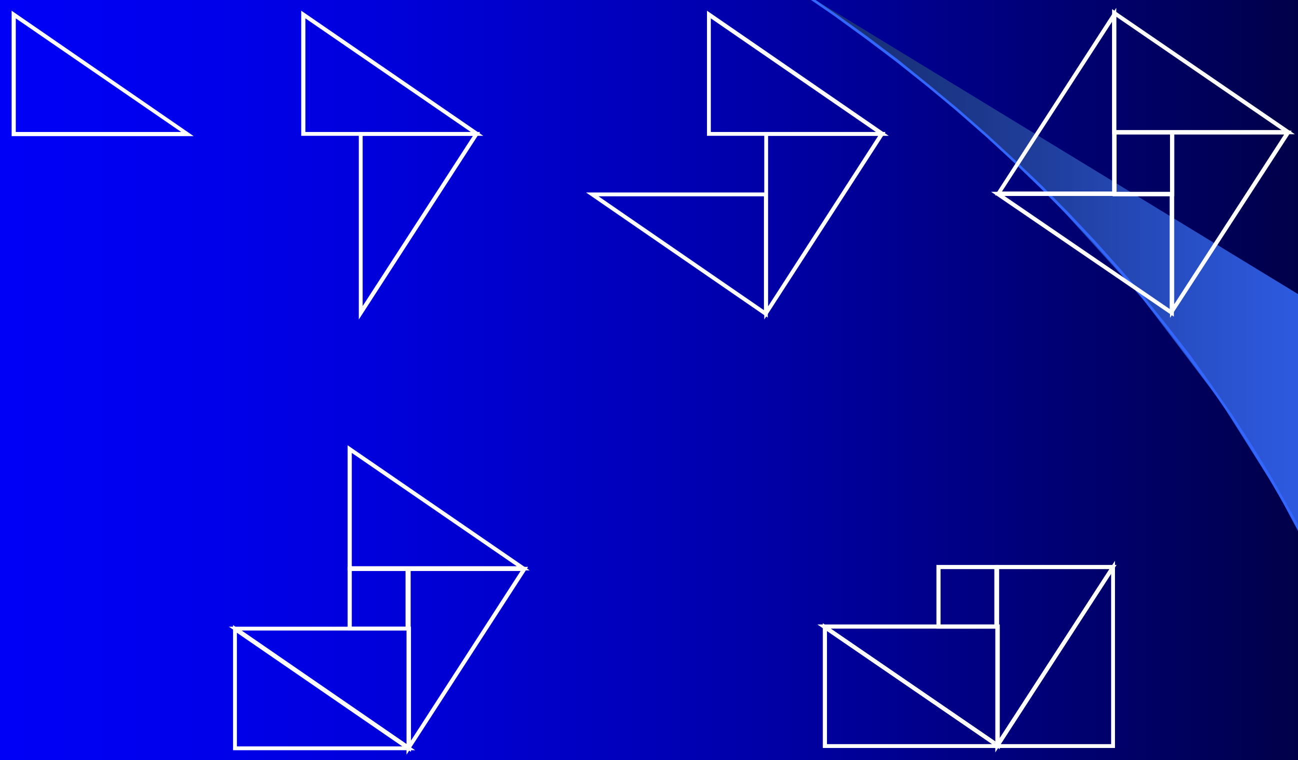
- What is their expressive power
- Why can they be interpreted so effectively
- What is the relation between logic and diagrammatic reasoning

Theorem of Pythagoras

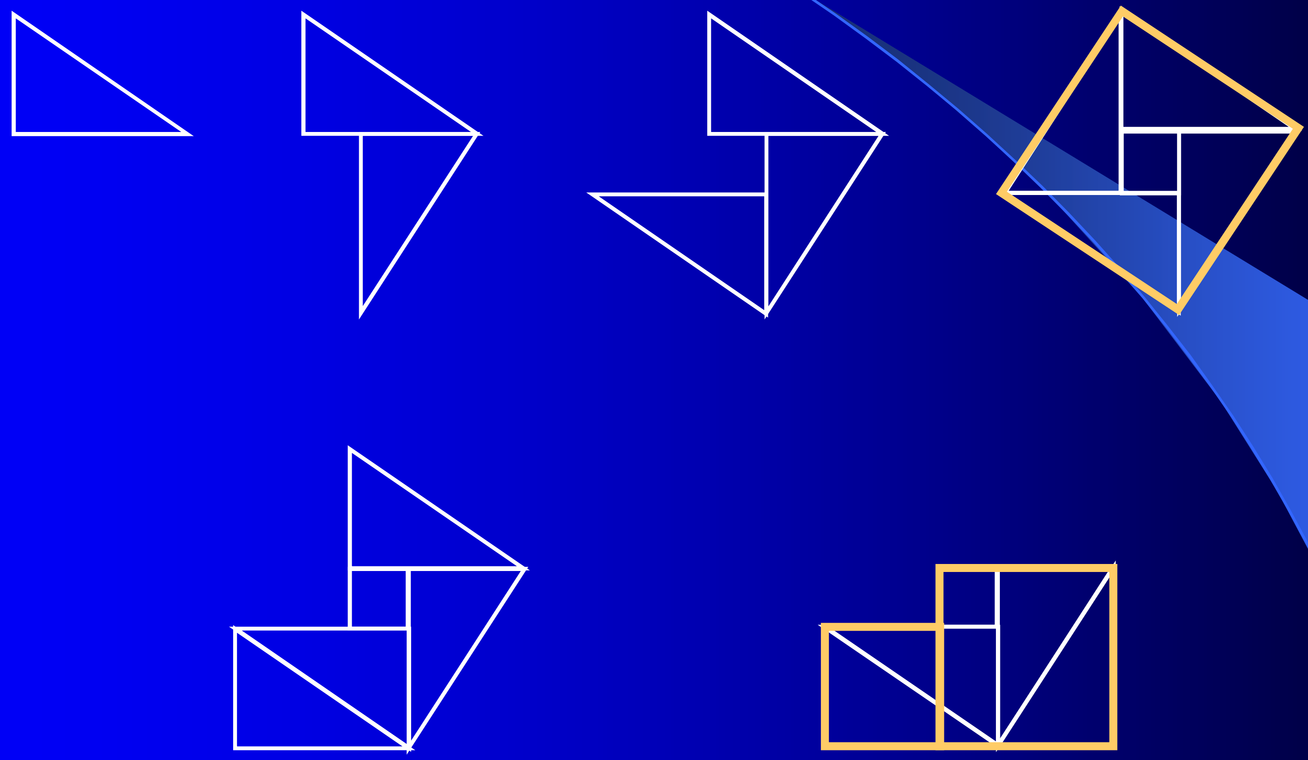


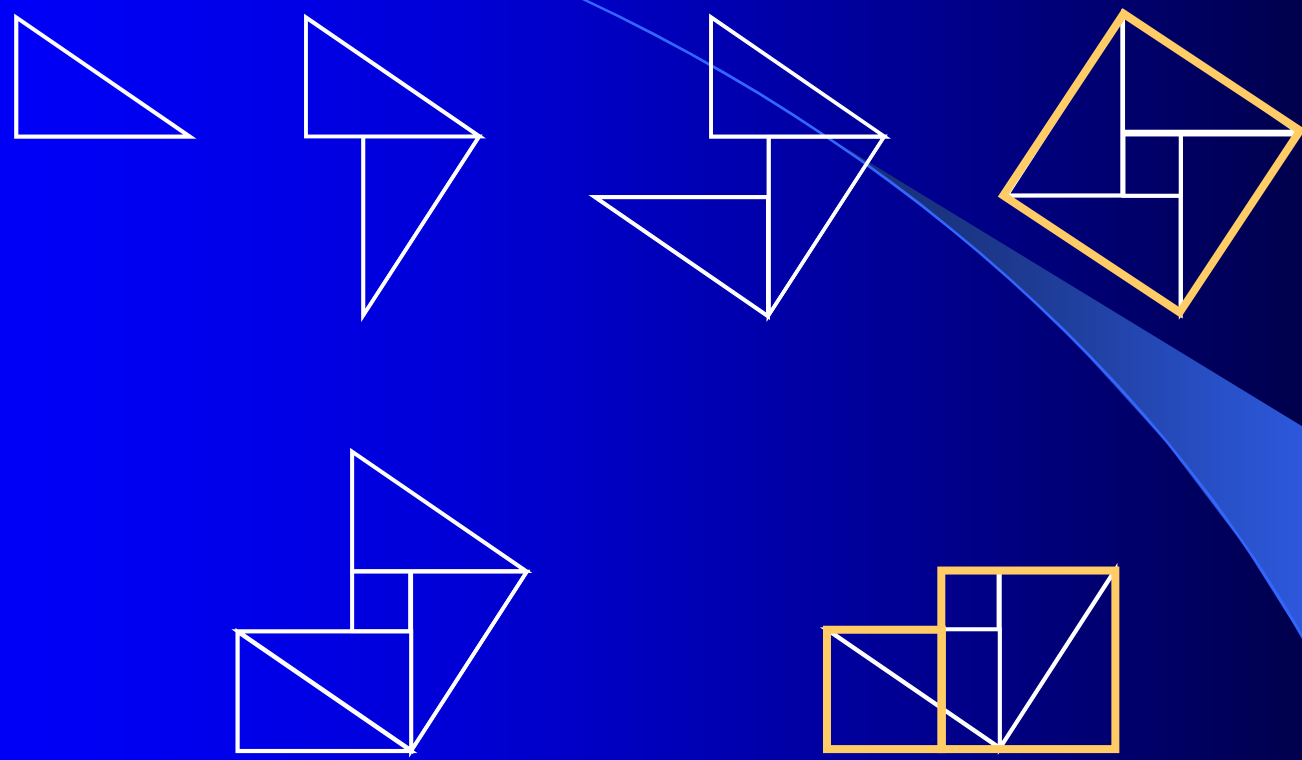
$$h^2 = a^2 + b^2$$

Bronowsky's proof



Bronowsky's proof





What are the mechanisms involved in this kind of reasoning?

A Challenge for AI

- Gelenter’s GTP (late 50’s): no account!
- Pineda (1989): The role of reinterpretations
- Barwise and Etchemendy: To illustrate heterogeneous reasoning (1990)
- Wang (1995): The need for generic descriptions
- Lindsay (1998): A demonstrator system
- Jamnik (1999): To illustrate a taxonomy of diagrammatic theorems

A Challenge for AI

- Pineda (2007):
  - A theory of diagrammatic reasoning
  - A semi-automatic proof of the theorem of Pythagoras
  - A semi-automatic proof of the theorem of the sum of the odds
  - A prototype program

The theory…

- Action schemes (a synthetic machinery)
- A notion of *re*-interpretation
- A geometric description machinery
- Conservation principles
- The arithmetic interpretation

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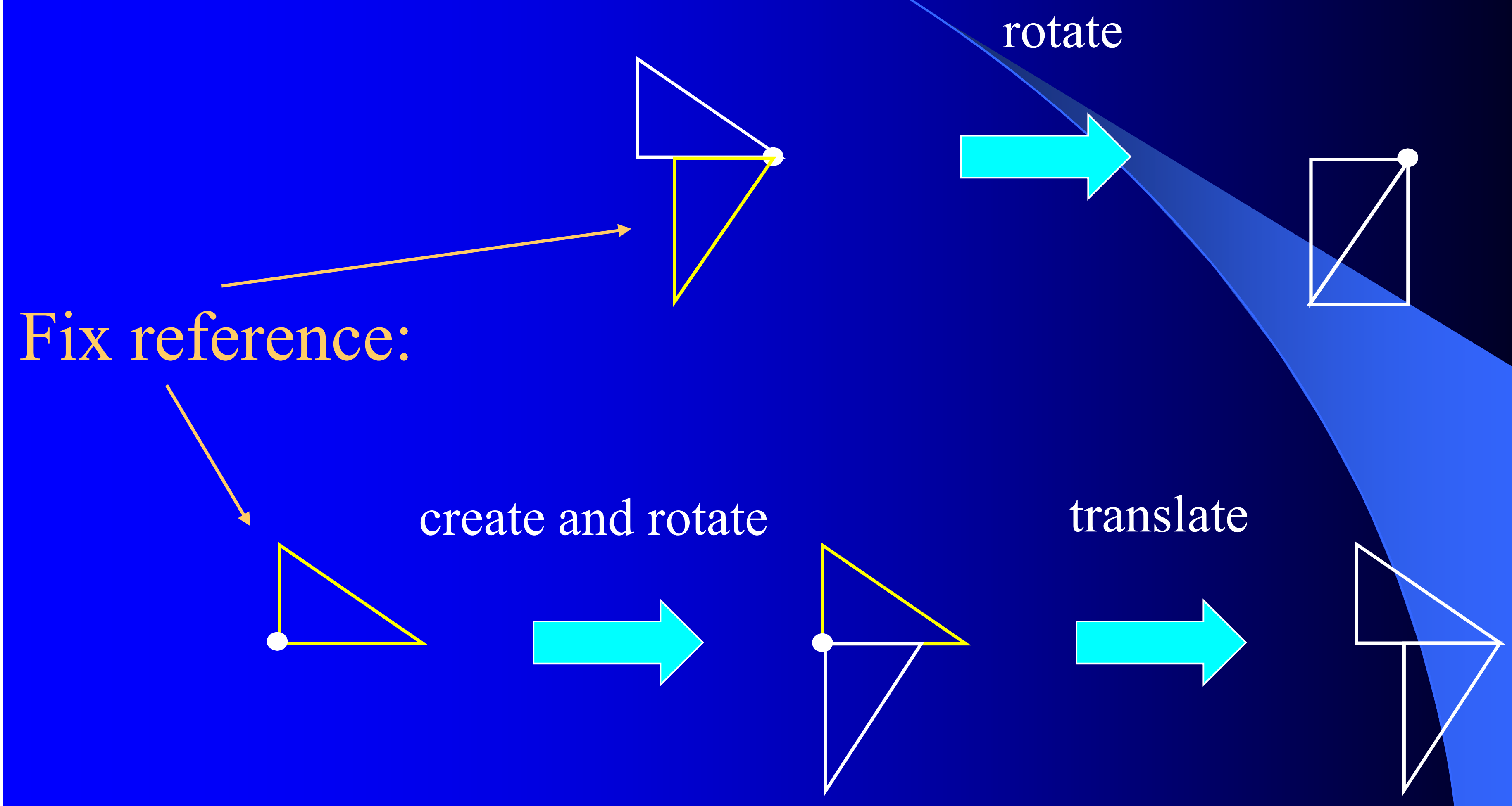
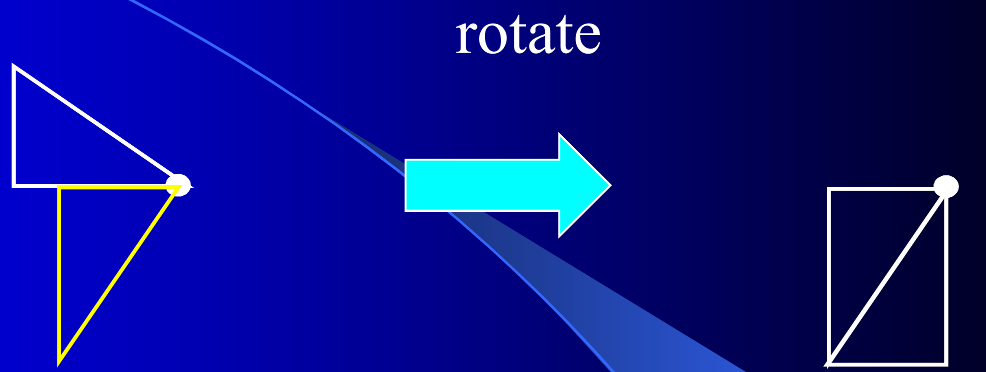
Action schemes

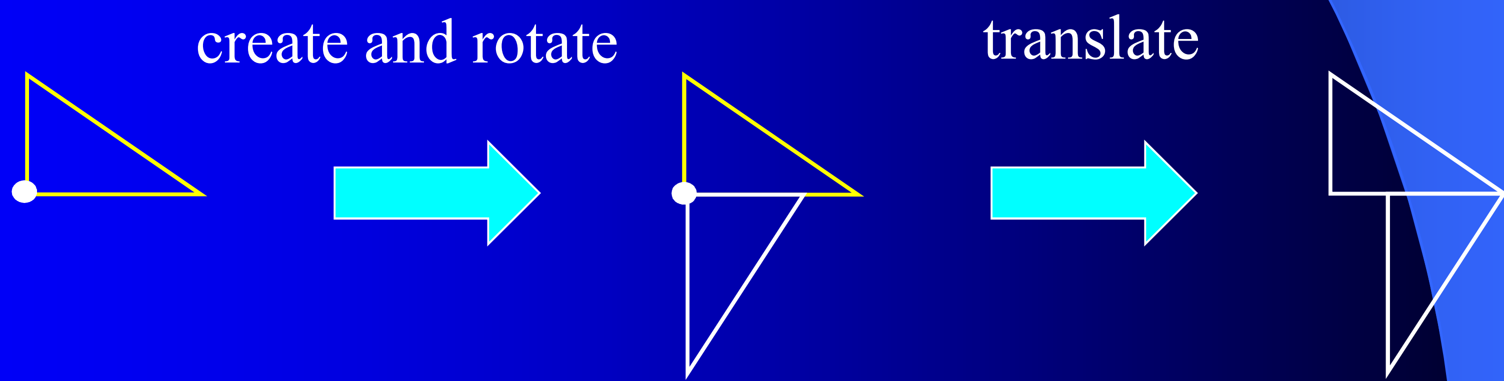


Preserving proprierties

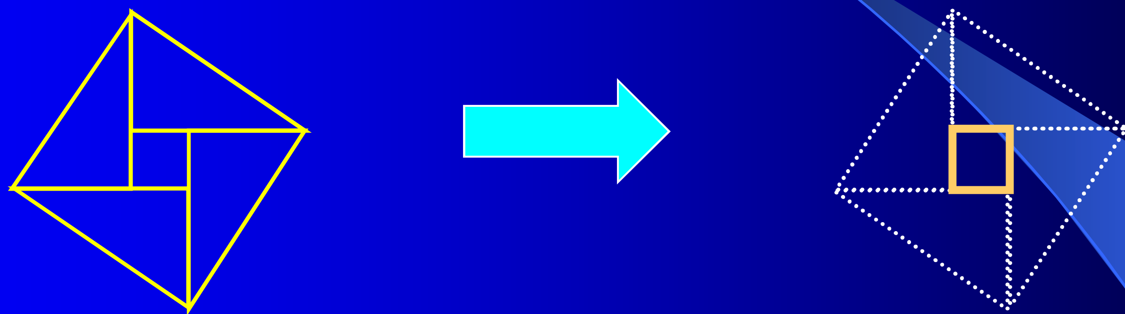
Area preserving:



Not area preserving:



A more complex action scheme



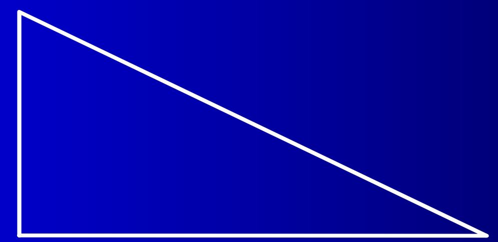
Not area preserving

Shape generation scheme

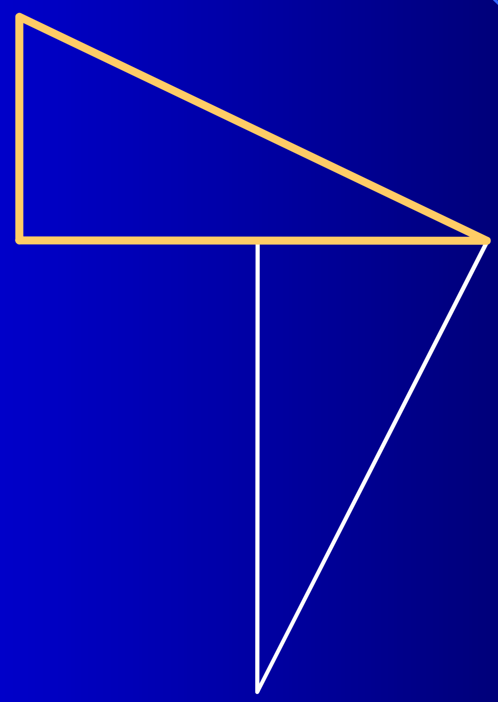
$diag_i, focus_i$ → $Action\ Scheme_j$ → $Diag_{i+1}, actee_j$

Heuristic's control and attention flow

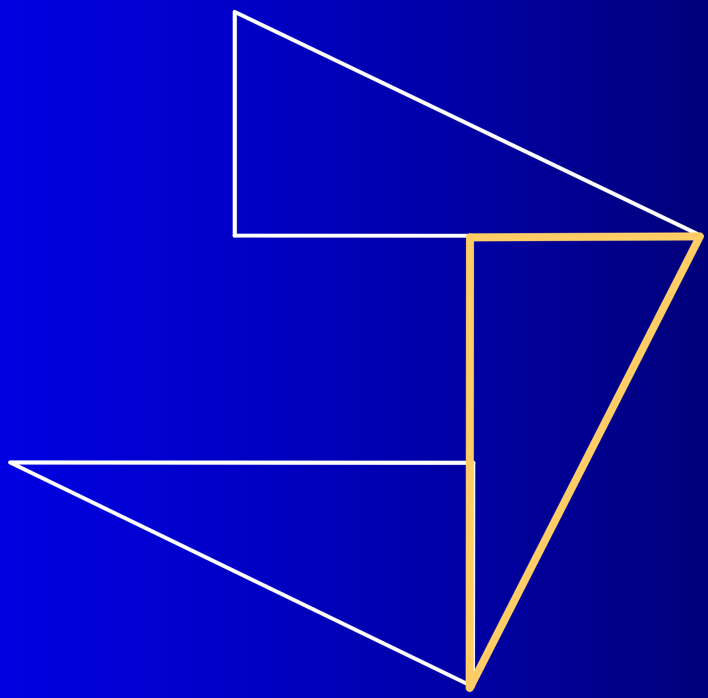
The right-triangle seed



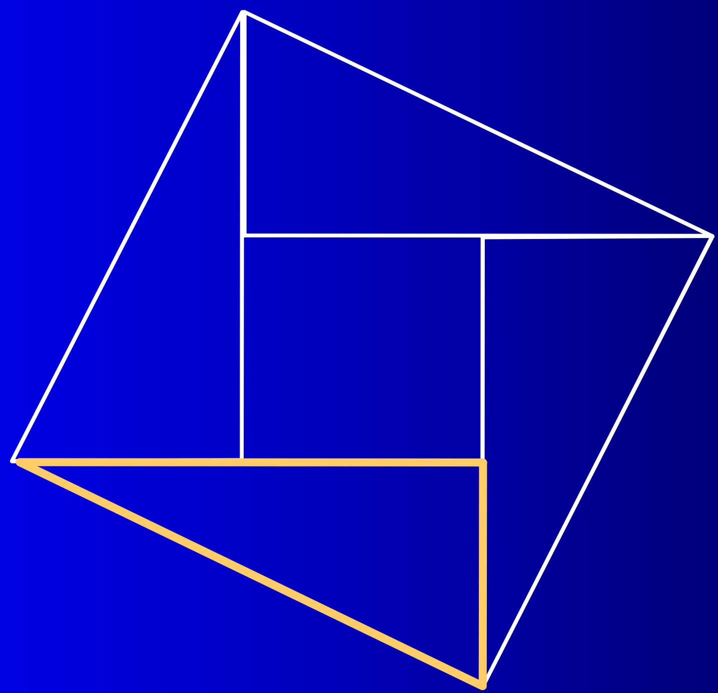
Application of scheme



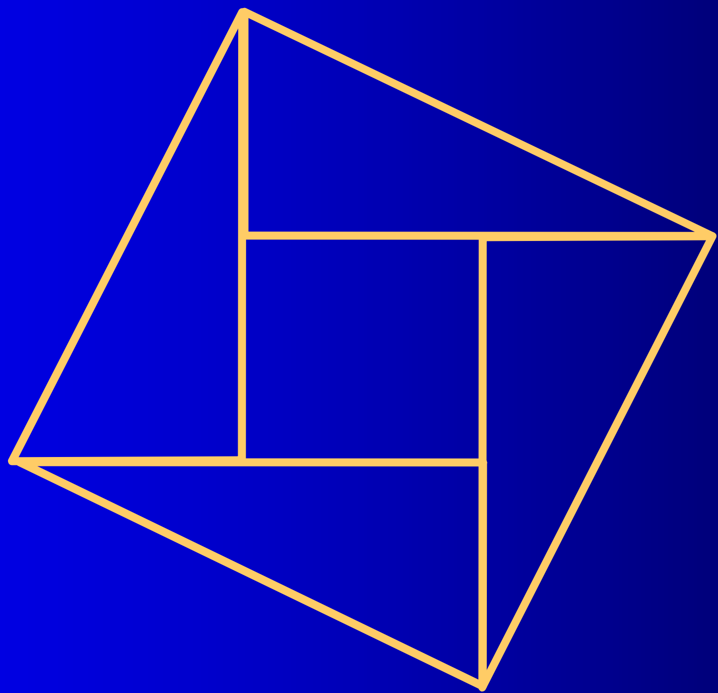
again…



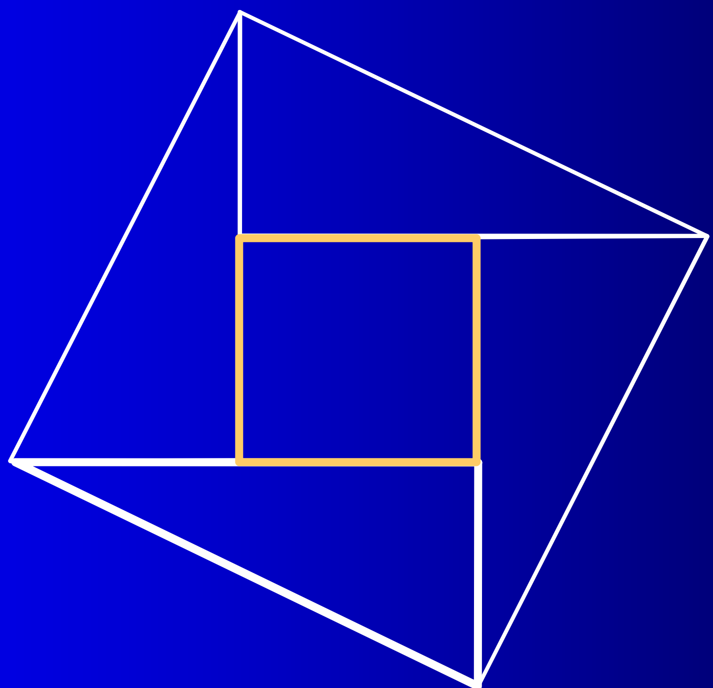
… and again!



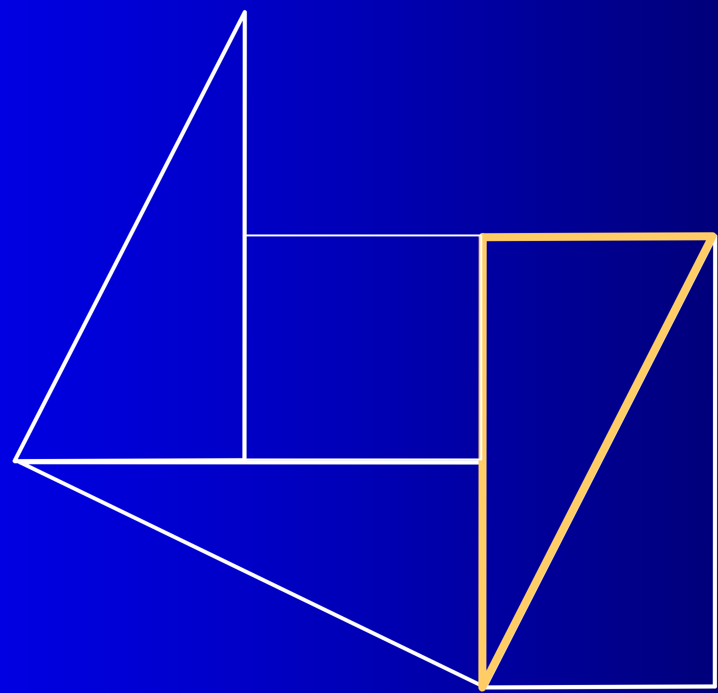
… A complex focus!



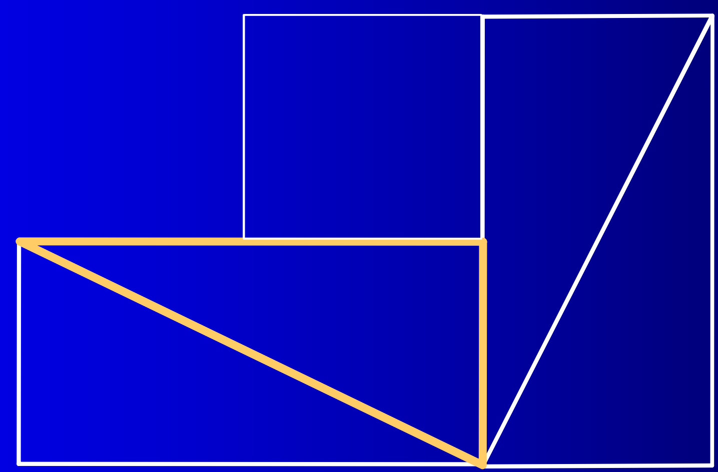
… Add internal square!



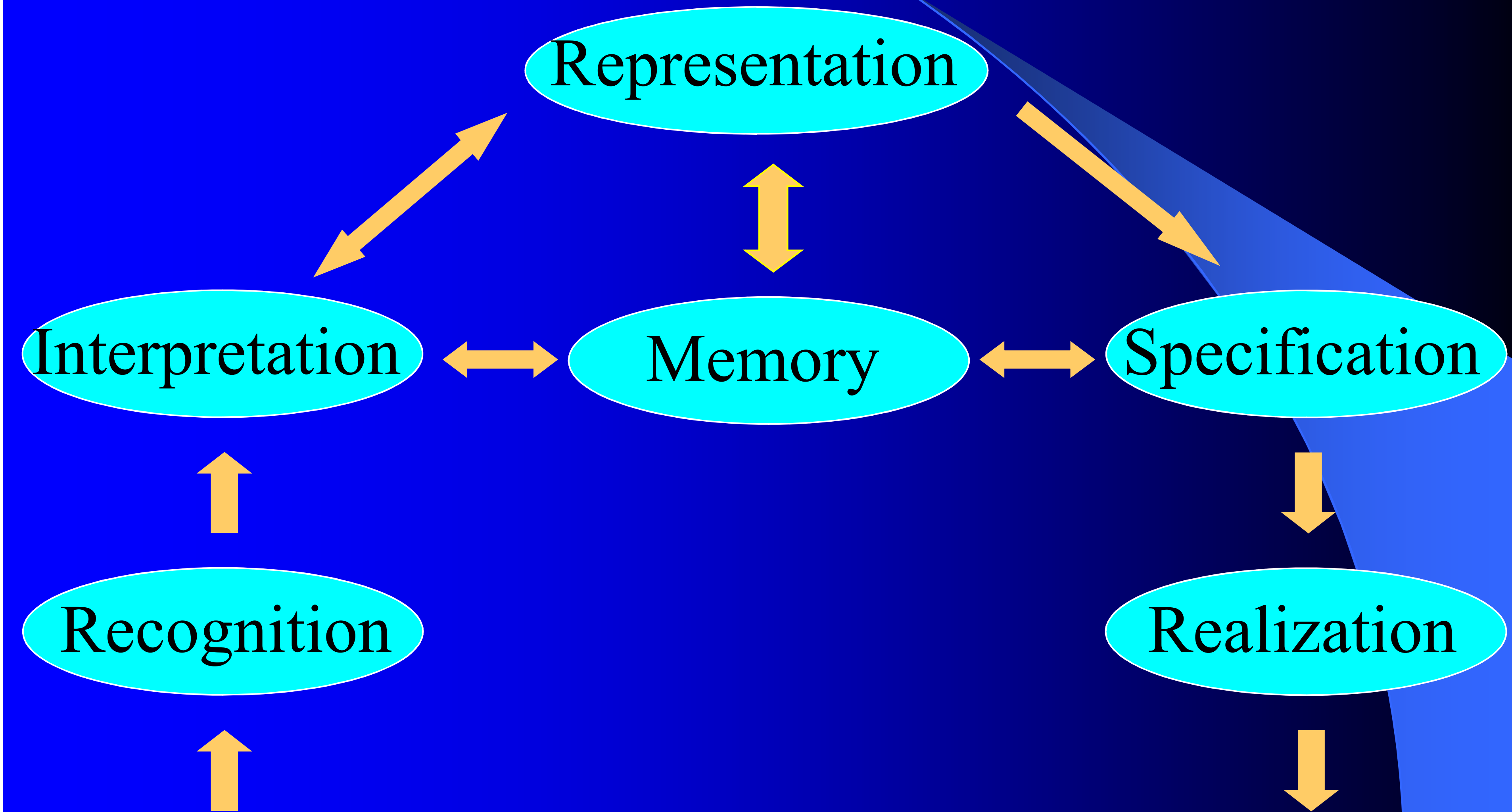
First area preserving transformation…



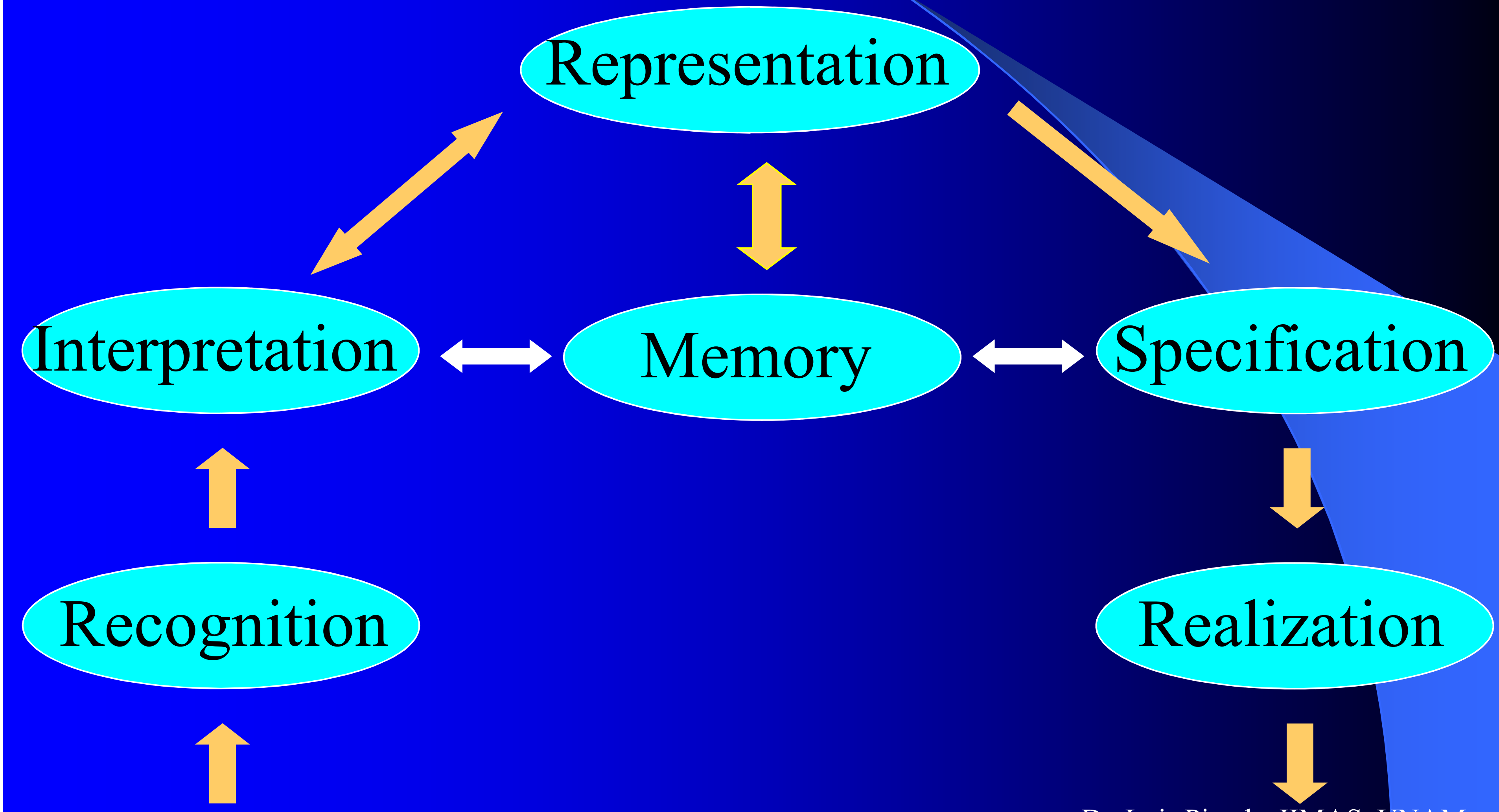
Second area preserving transformation



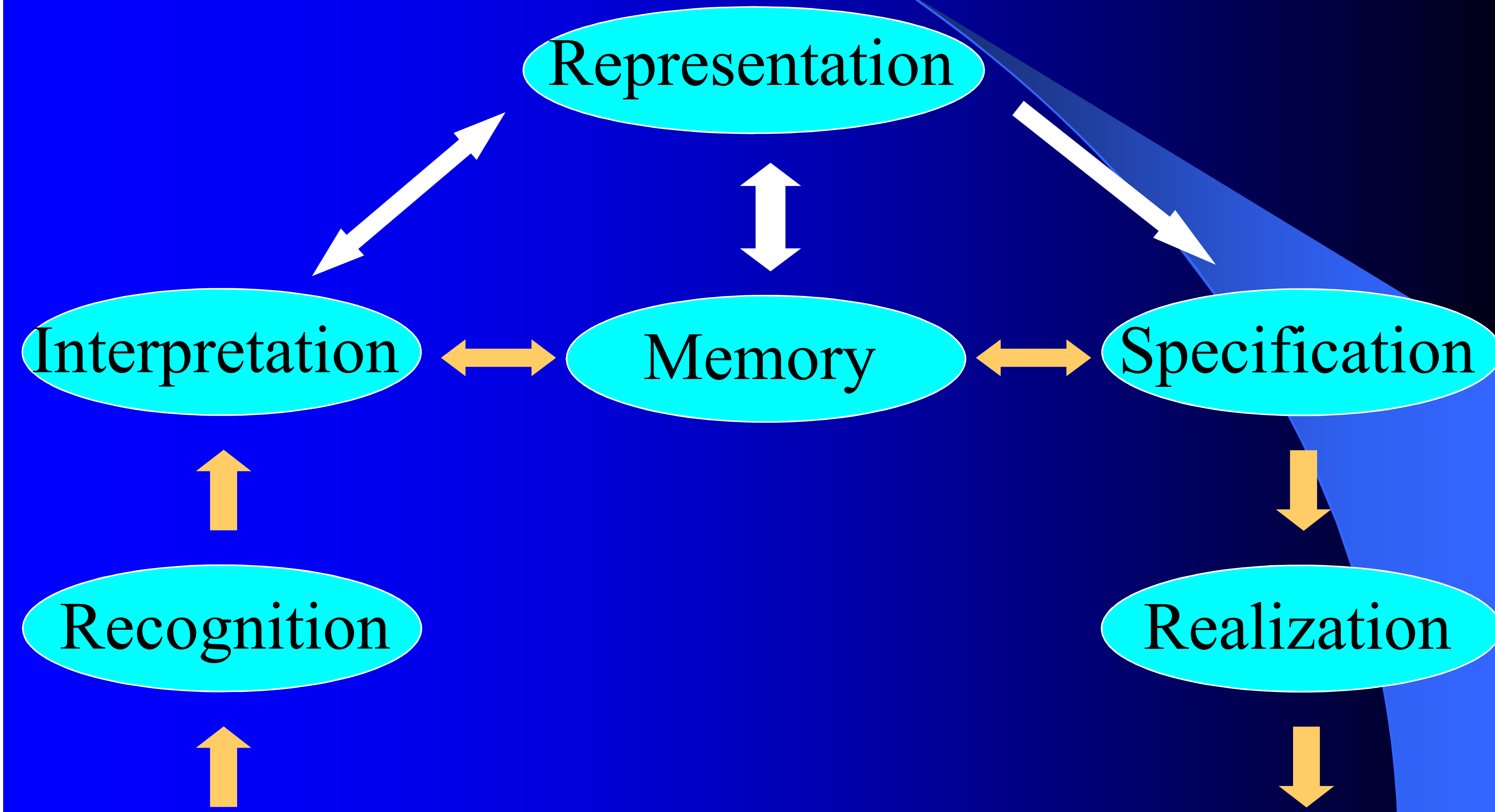
Diagramatic Reasoning: action schemes



Schemes 1 and 2



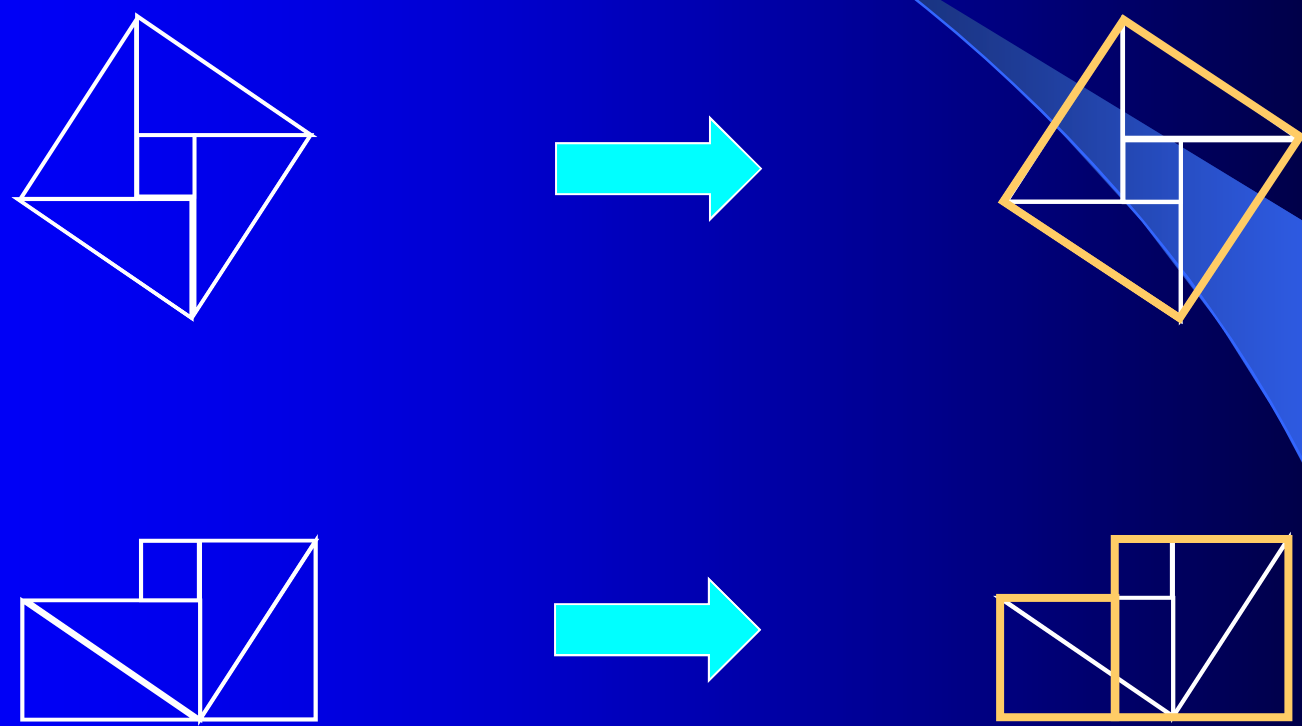
Scheme 3


Representation
Interpretation
Memory
Specification
Recognition
Realization

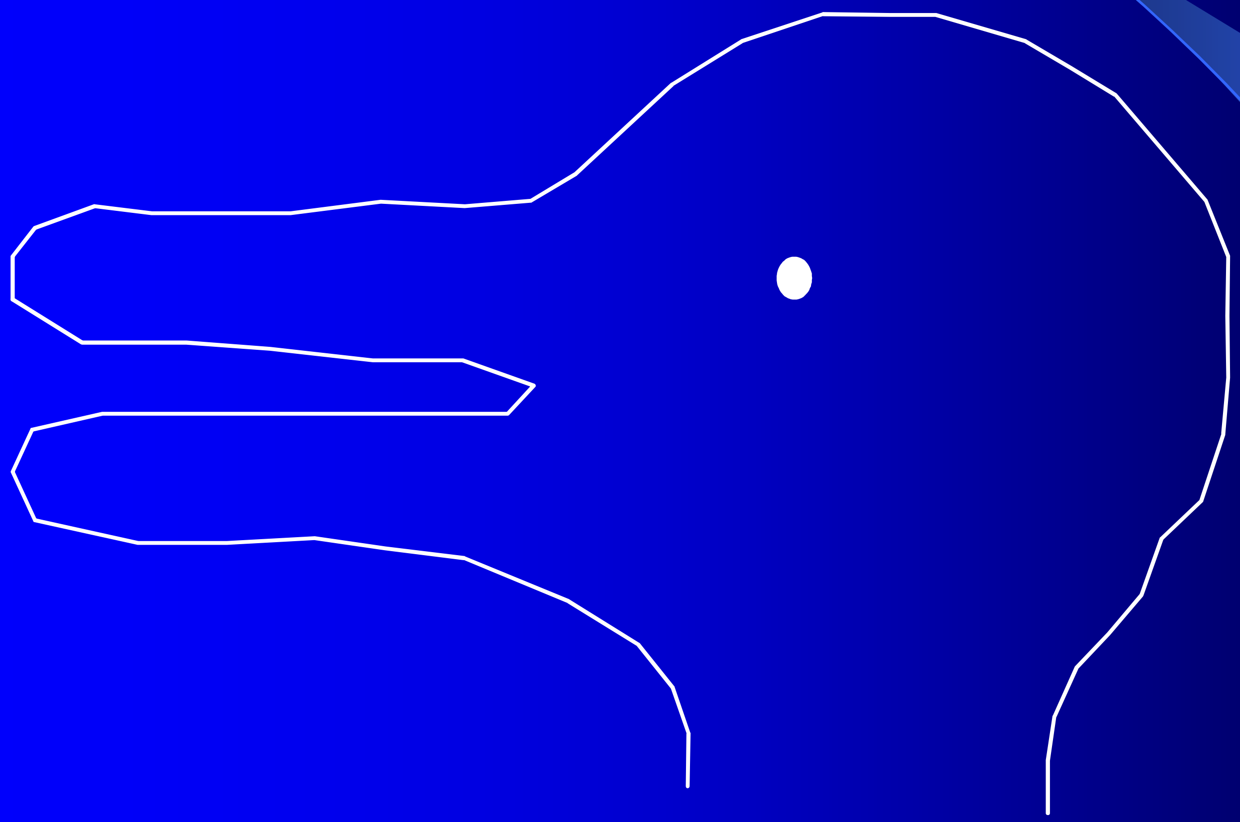
The theory…

- Action schemes (a synthetic machinery)
- A notion of *re*-interpretation
- A geometric description machinery
- Conservation principles
- The arithmetic interpretation

The *re*-interpretations and "emerging" objects



A change in the conceptual perspective!

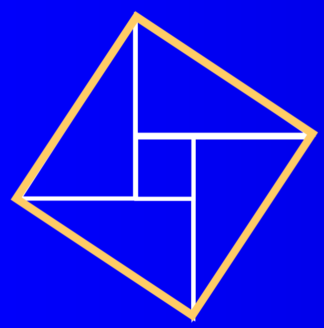


A problem of description…

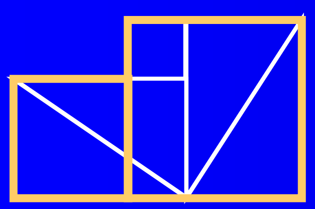
diagram → *Perceptual Scheme* → *Description*

A perceptual inference?

We need the relevant description

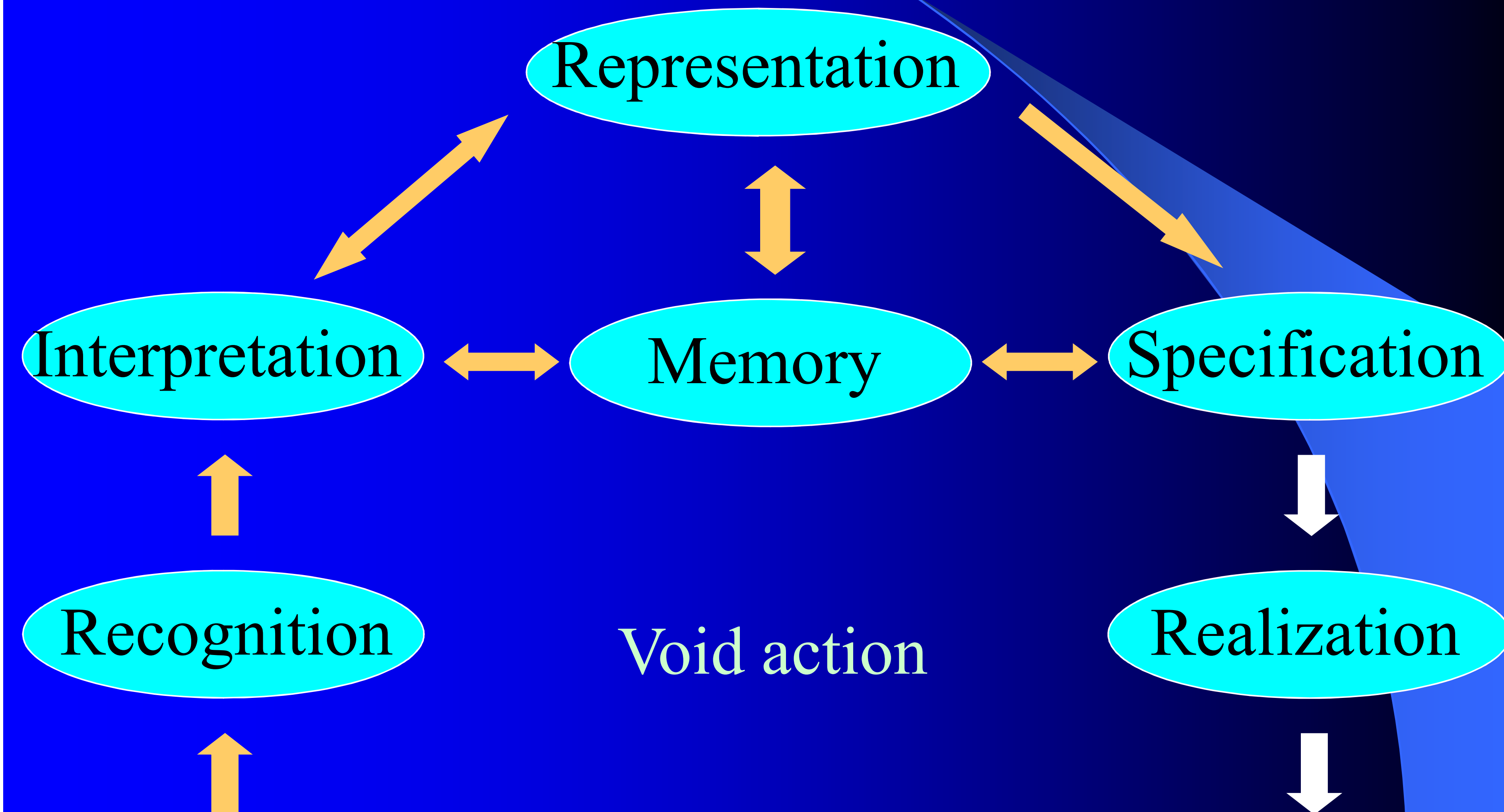


A square on the hypotenuse of a right-triangle



The union of a square on a right side of a right triangle and a square on the other right side of the same right triangle

Diagramatic Reasoning: Perceptual Inference



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Concepts

- Concepts (i.e. knowlegde objects) can be represented in computers
- Turing Machines campute functions
- So, concepts are represented through functions
- The challange is to find such functions
- In the present case, the functions representing geometric and arithmetic concepts that are expressed through diagrams!

Geometric description machinery

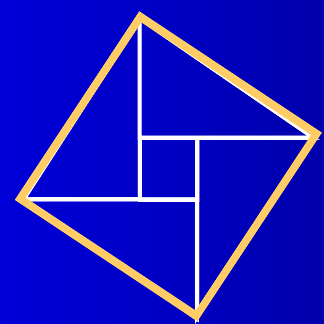
- A geometric signature to refer to geometric objects, properties and relations
- The functional abstractor operator to express geometric concepts
- A geometric descriptor operator to refer to (contextually dependent) emerging objects:

$$T <= f$$

  - If $f(A)$ is true $(T <= f) = T$ where T is a term of any geometric sort which contains (possible) variables in f

Generic description

- Diagram:

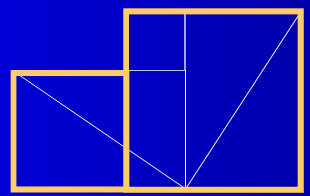


- Description: $y <= f_1$
- where:

$$f_1 = \lambda x \lambda y.right\_triang(x)\ \&\ \text{square}(y)\ \&\ side(hipotenuse(x), y)$$

Generic description

- Diagram:



- Description:

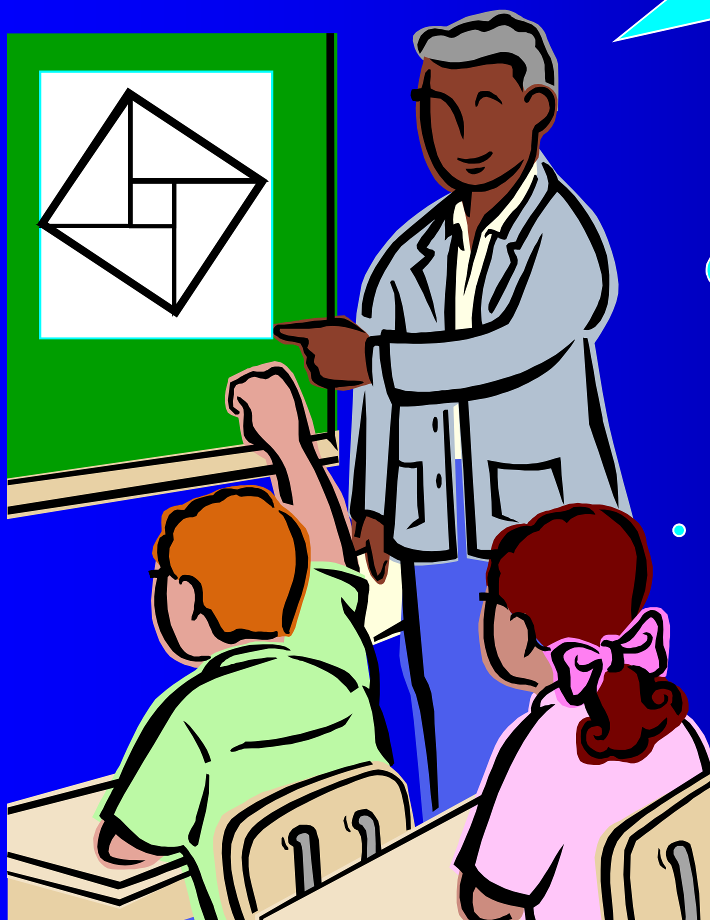
  $union(y, z) <= f_2$

  where:

  $f_2 = \lambda x \lambda y \lambda z. right\_triang(x)\ \&\ square(y)\ \&\ square(z)\ \&\ side(side\_a(x), y)\ \&\ side(side\_b(x), z)$

Diagrams and descriptions

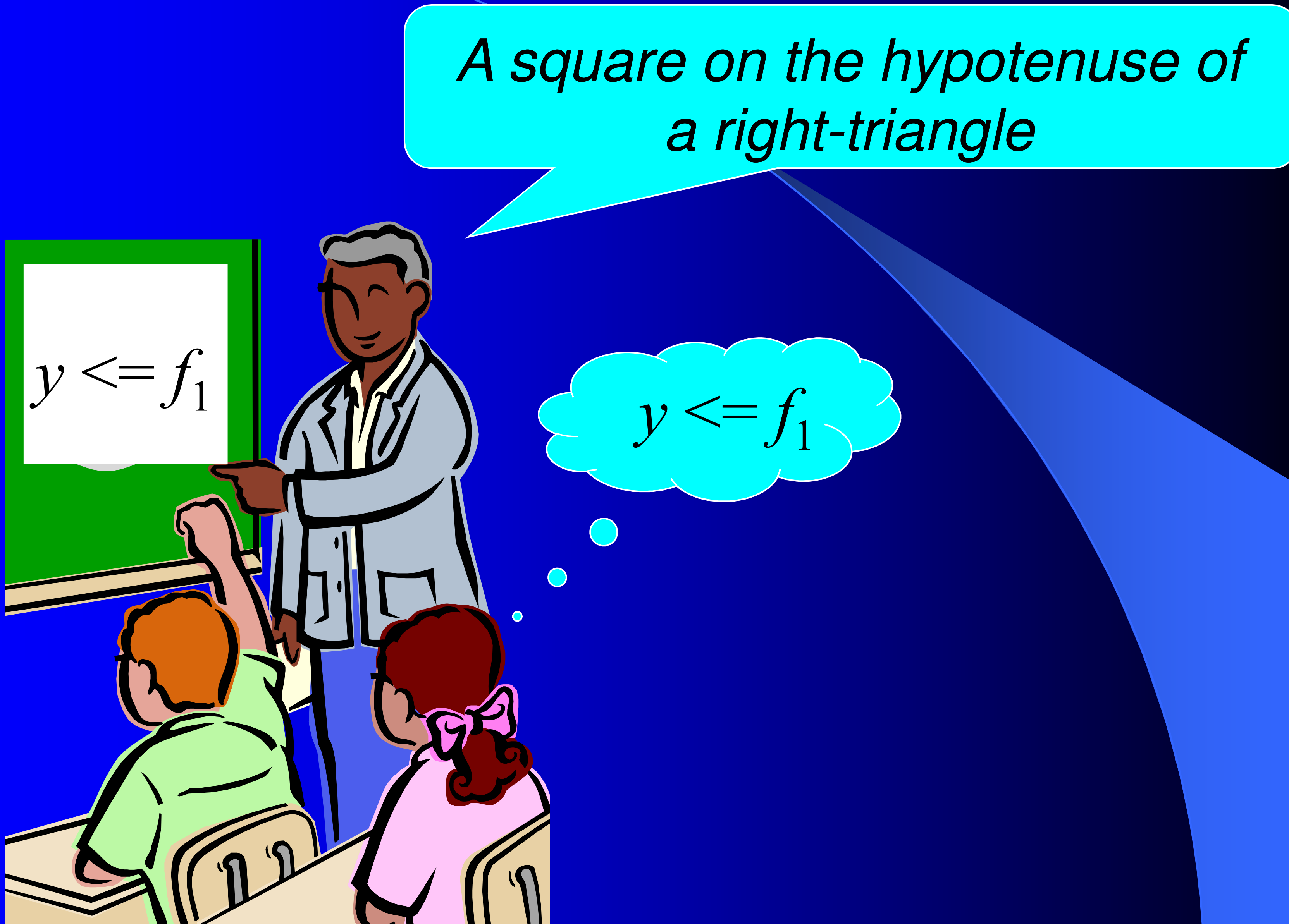
A square on the hypotenuse of a right-triangle



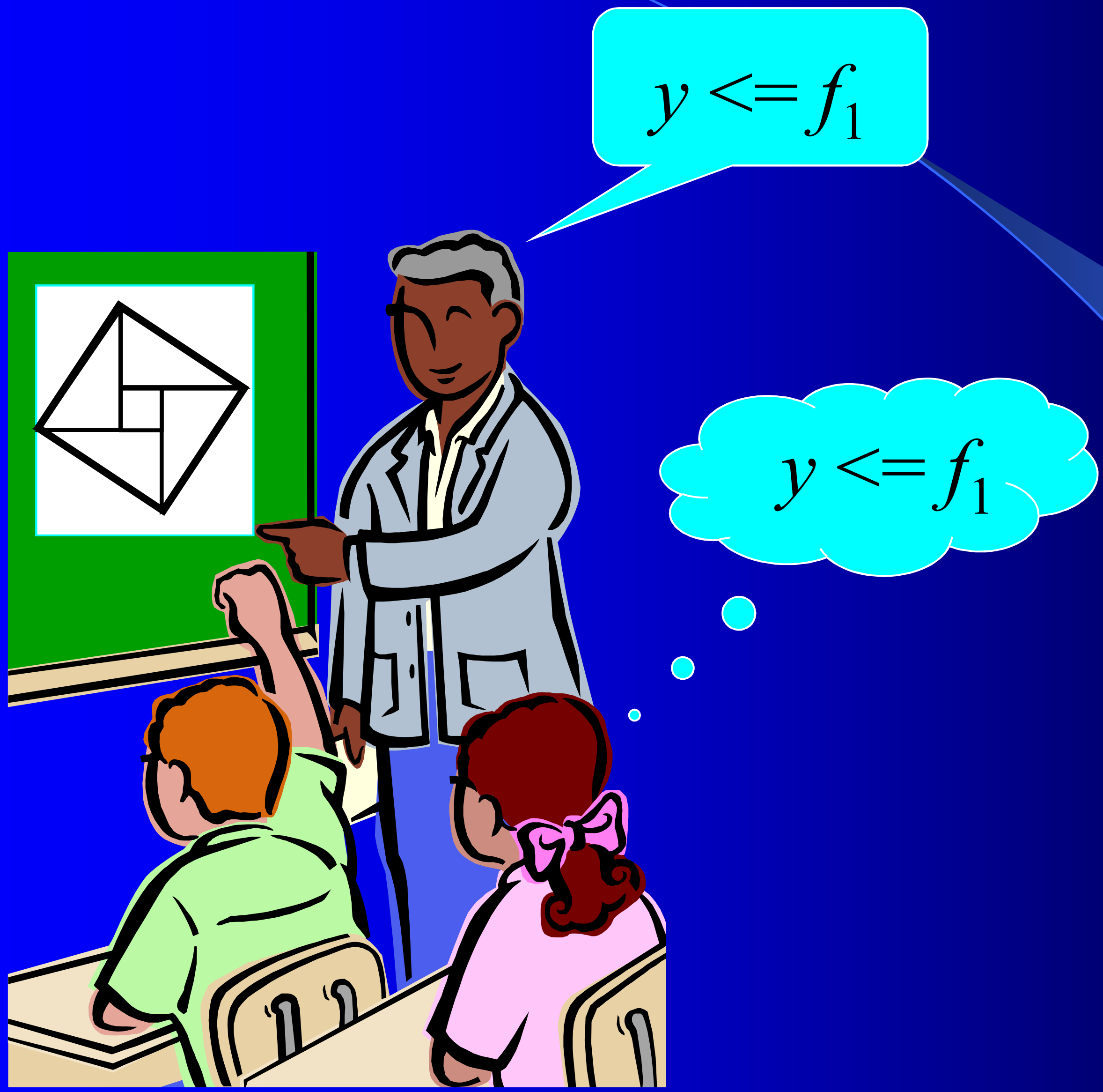
$$y <= f_1$$

Descriptions as internal Representations?

Diagrams and descriptions

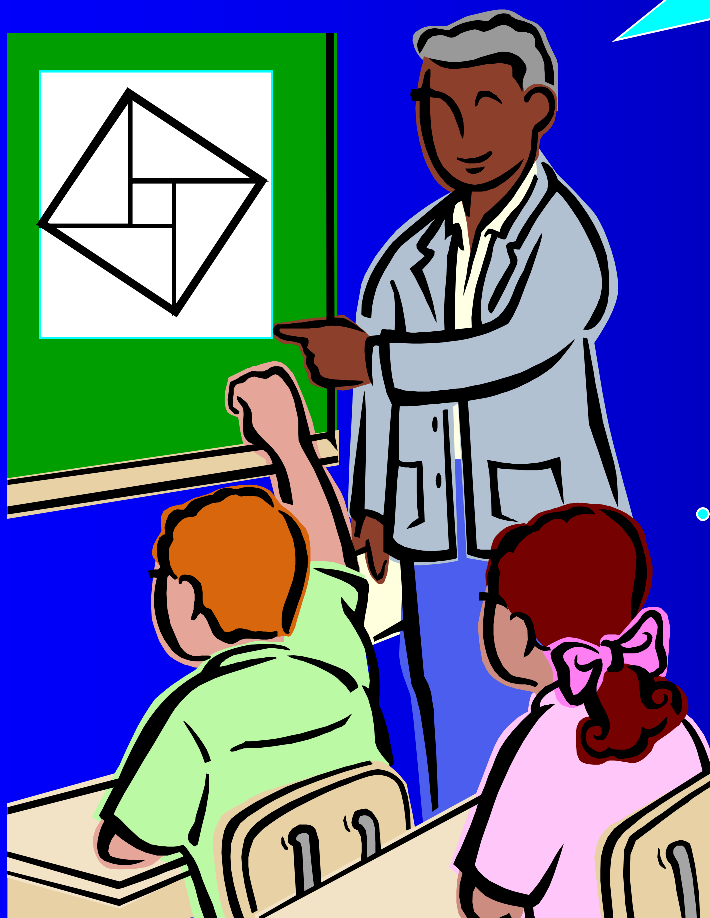


Diagrams and descriptions



Functions represent meanings!

A square on the hypotenuse of a right-triangle



Meaning

$y <= f_1$ represents a generic concept!

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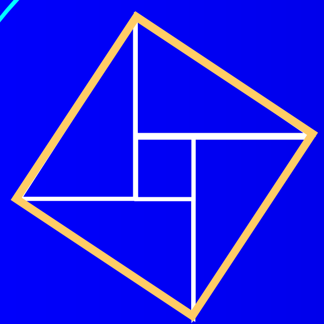
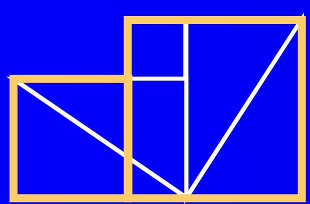
We need to state a property holds for different diagrams…

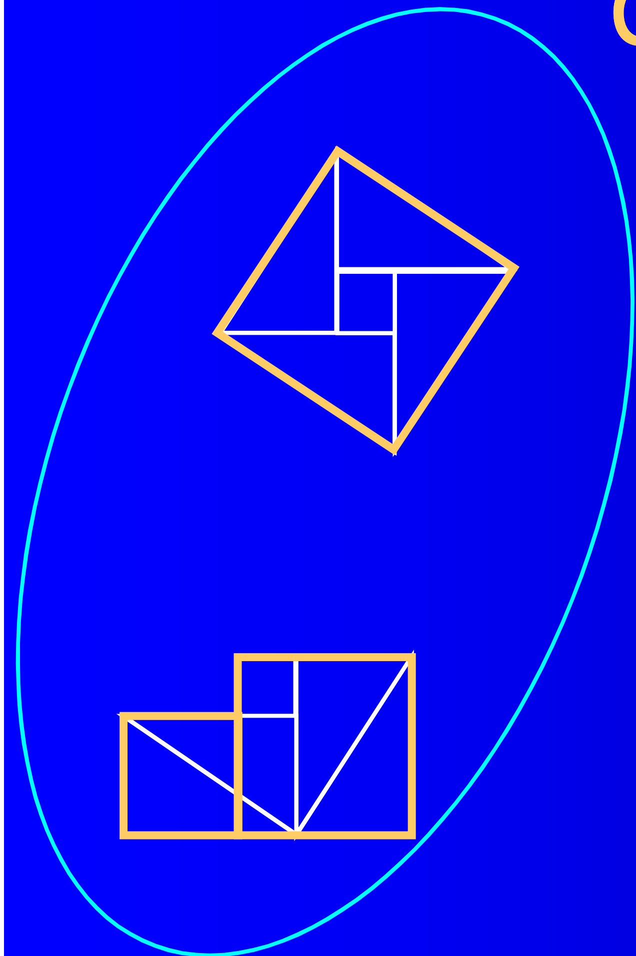


Area of a square on the hypotenuse of a right-triangle



Area of the union of a square on a right side of a right triangle and a square on the other right side of the same right triangle

This is a relation between generic descriptions…



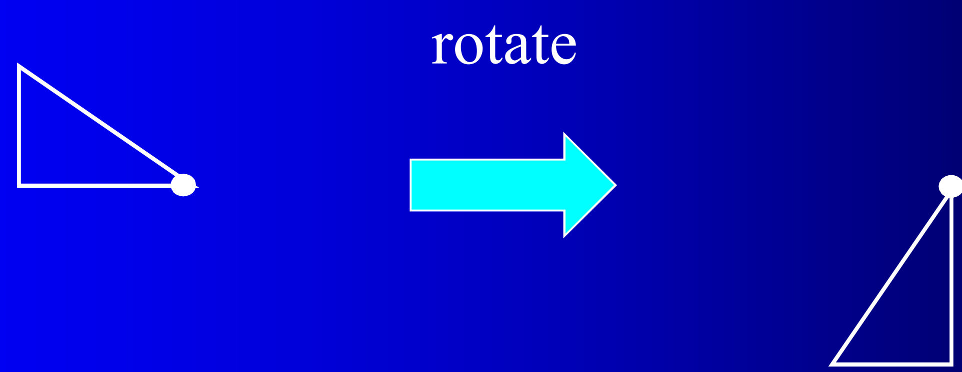
$$area(y <= f_1)$$

$$area(union(y, z) <= f_2)$$

Conservation principles

- Generalized concept of equality for geometrical properties
- <u>Global</u> principle of conservation of area:

$$\lambda P \lambda Q(area(P) = area(Q))$$



- The application of the principle is granted if the action scheme producing the transformation preserves the conservation property

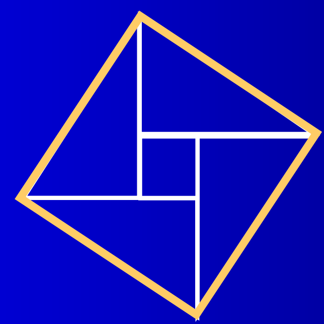
Conservation principles

- Structured principle of conservation of area:

$$\lambda P \lambda Q \lambda x (area(P(x)) = area(Q(x)))$$

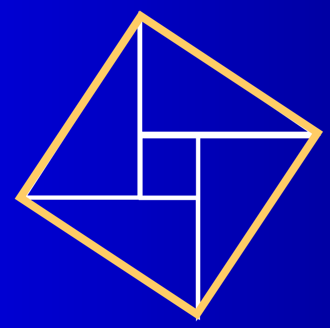
  - P and Q are generic descriptions of geometrical objects or configurations
  - x is a generic reference object
- An interpretation act (under the appropriate conditions) is represented by a functional application operation!

Synthesis of geometric concepts



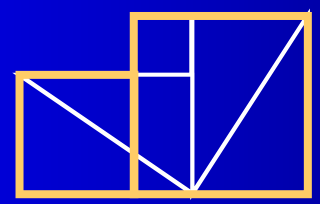
$$\lambda P \lambda Q \lambda x (area(P(x)) = area(Q(x)))(y <= f_1)$$

Synthesis of geometric concepts



$$\lambda Q \lambda x(area(y <= f_1(x)) = area(Q(x))$$

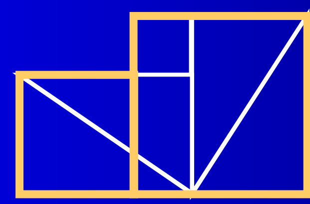
Synthesis of geometric concepts



$$\lambda Q \lambda x(area(y <= f_1(x)) = area(Q(x))(union(y, z) <= f_2)$$

The application is permitted if the the diagram is modified by an area preserving (sequence of) transformation

Synthesis of geometric concepts



$$\lambda x(area(y <= f_1(x)) = area(union(y, z) <= f_2(x))$$

The function representing the geometric concept of the Theorem of Pythagoras!

The geometric concept

$$f_{TP} = \lambda x(area((w <= f_1)(x)) = area((union(y, z) <= f_2)(x))$$

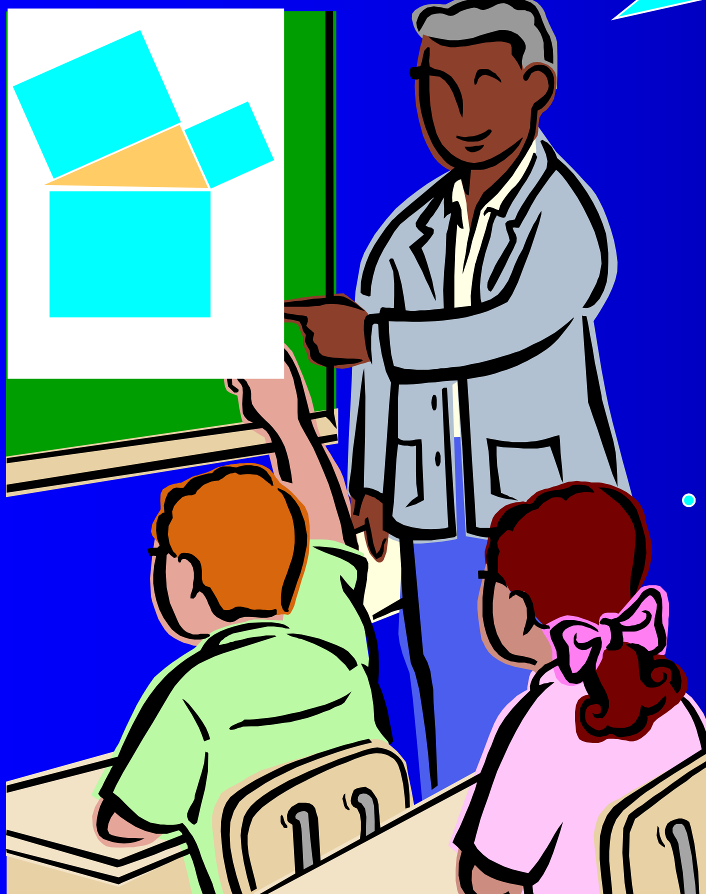
Where:

$$f_1 = \lambda x \lambda y.right\_triang(x) \ \& \ \text{square}(y) \ \& \ side(hipotenuse(x), y)$$

$$f_2 = \lambda x \lambda y \lambda z.right\_triang(x) \ \& \ square(y) \ \& \ square(z) \ \& \ side(side\_a(x), y) \ \& \ side(side\_b(x), z)$$

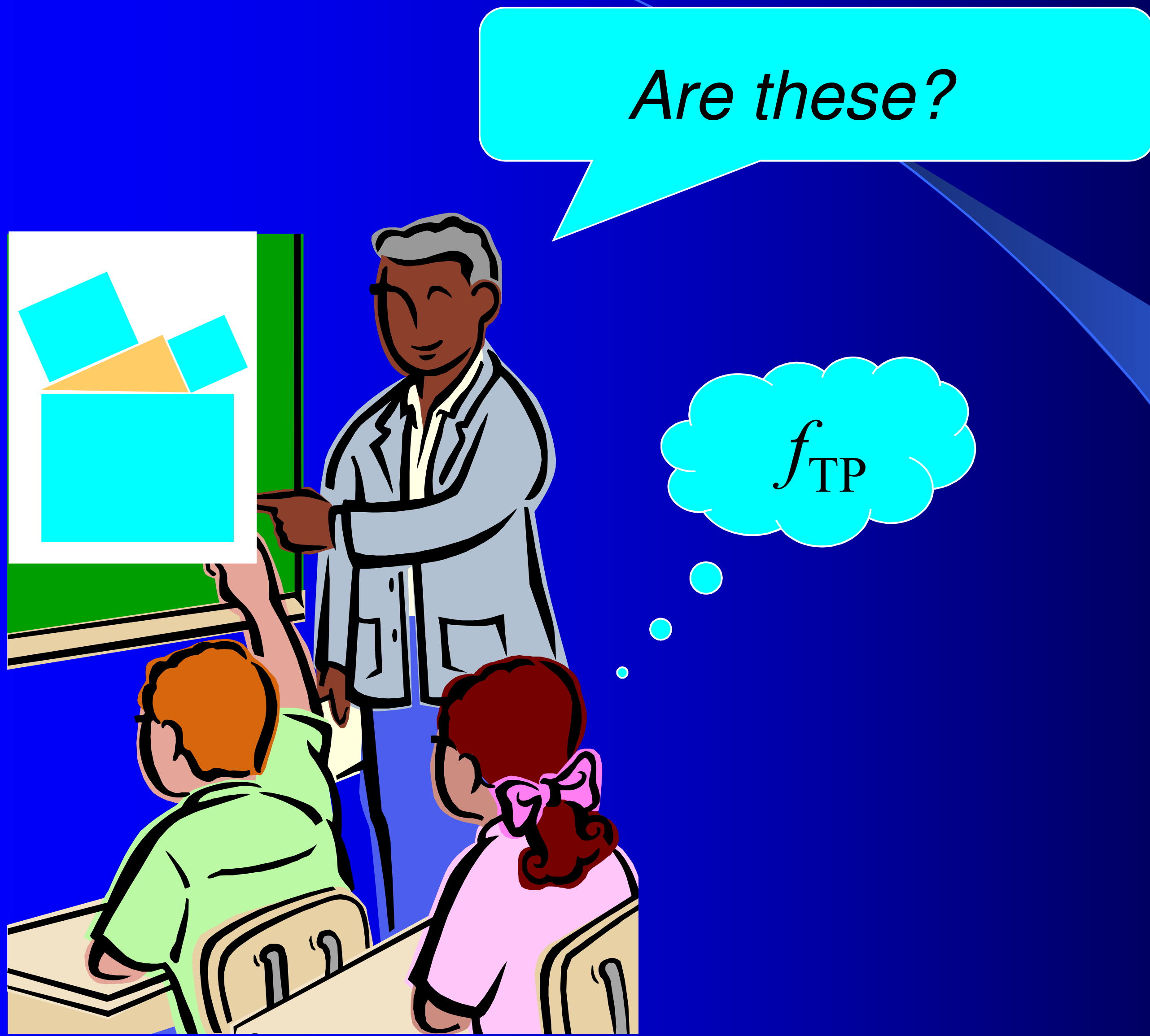
The extension of the concept…

Are these in the Pythagorean relation?



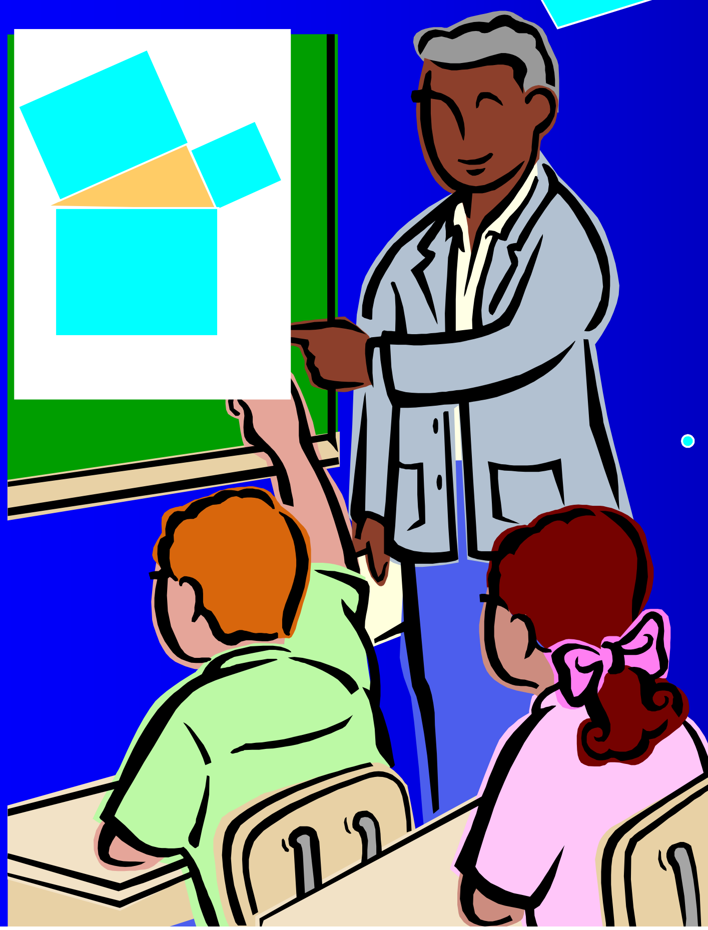
f_{TP}

The extension of the concept…



Representation of meanings!

The area on the hypotenuse of a right triangle is the same as the area of the union of the squares on its right sides



f_{TP}

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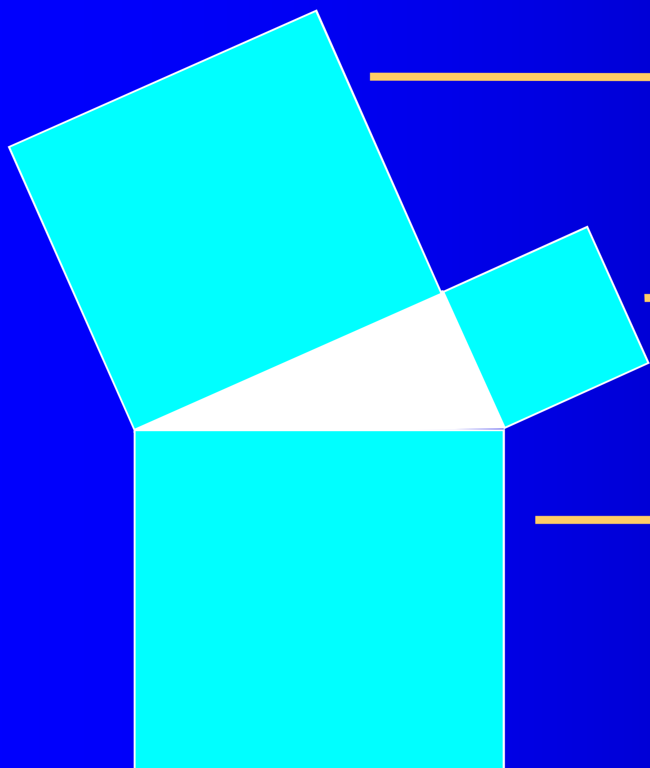
A mapping from the geometry into the arithmetic



$$h^2 = a^2 + b^2$$

The representation function

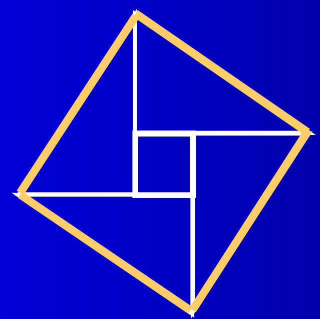
$\phi(x <= f) = \lambda u.u^2$ if the type of x *in* f is sq

$\phi(union) = +$

$\phi(g(y_1, y_2) <= f) = \phi(g)(\phi(y_1 <= f), \phi(y_2 <= f))$

The mapping

Diagram:

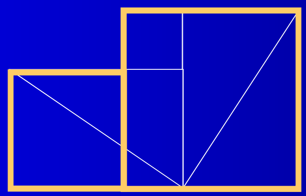


The arithmetic concept:

$$\phi(y <= f_1) = \lambda u.u^2$$

The mapping…

The diagram:



The arithmetic concept:

$$\phi(union(y, z) <= f_2) = \lambda v.v^2 + \lambda w.w^2$$

The mapping

- The geometric principle:
  - $\lambda P \lambda Q \lambda x(area(P(x)) = area(Q(x))$
- The arithmetic principle:
  - $\lambda P \lambda Q(P = Q)$
  - Concept of global aritmetic equality!

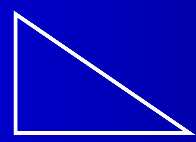
Diagrammatic Derivations

A three-tier tandem process

- The synthesis of geometric form
- The synthesis of the geometric concept
- The synthesis of the arithmetic concept

The seed...

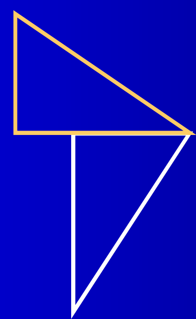
- Diagram:



- Principle of conservation of area:
  - $\lambda P \lambda Q \lambda x(area(P(x)) = area(Q(x))$
- Concept of the global arithmetic equality:
  - $\lambda P \lambda Q(P = Q)$

Synthesis of form

- Diagram:



- Principle of conservation of area:
  - $\lambda P \lambda Q \lambda x(area(P(x)) = area(Q(x))$
- Concept of the global arithmetic equality:
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Synthesis of form

- Diagram:

- Principle of conservation of area:
  - $\lambda P \lambda Q \lambda x(area(P(x)) = area(Q(x))$
- Concept of the global arithmetic equality:
  - $\lambda P \lambda Q(P = Q)$

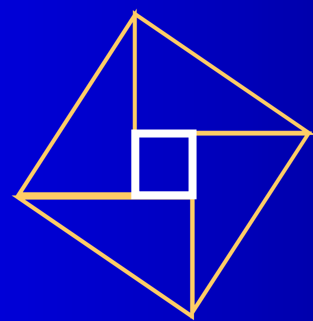
Synthesis of form

- Diagram:

- Principle of conservation of area:
  - $\lambda P \lambda Q \lambda x(area(P(x)) = area(Q(x)))$
- Concept of the global arithmetic equality:
  - $\lambda P \lambda Q(P = Q)$

Synthesis of form

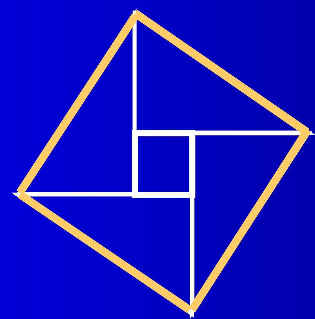
- Diagram:



- Principle of conservation of area:
  - $\lambda P \lambda Q \lambda x(area(P(x)) = area(Q(x))$
- Concept of the global arithmetic equality:
  - $\lambda P \lambda Q(P = Q)$

First reinterpretation

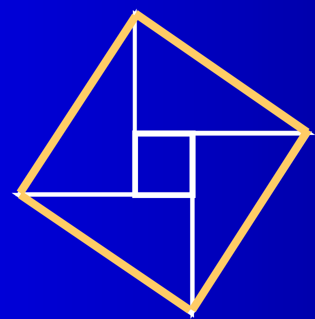
- Reinterpretations preserve area:



- Concepts construction:
  - $\lambda P \lambda Q \lambda x(area(P(x)) = area(Q(x))(w <= f_1)$
  - $\lambda P \lambda Q(P = Q)(\lambda u.u^2)$

First reinterpretation

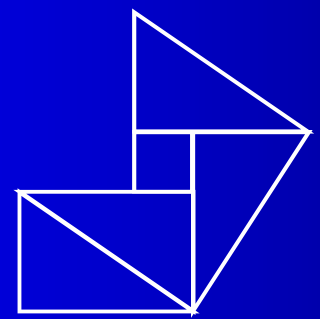
- Reinterpretation:



- Concepts construction:
  - $\lambda Q \lambda x(area(w <= f_1(x)) = area(Q(x))$
  - $\lambda Q(\lambda u.u^2 = Q)$

Synthesis of form

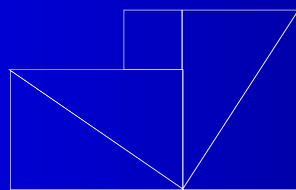
- Diagram:



- Concepts construction:
  - $\lambda Q \lambda x(area(w <= f_1(x)) = area(Q(x))$
  - $\lambda Q(\lambda u.u^2 = Q)$

Synthesis of form

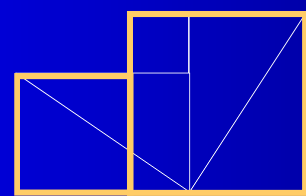
- Diagram:



- Concepts construction:
  - $\lambda Q \lambda x(area(w <= f_1(x)) = area(Q(x))$
  - $\lambda Q(\lambda u.u^2 = Q)$

Second reinterpretation

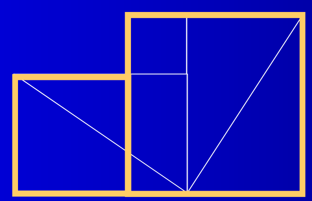
- Reinterpretation:



- Concepts construction:
  - $\lambda Q \lambda x(area(w <= f_1(x)) = area(Q(x))(union(y, z) <= f_2)$
  - $\lambda Q(\lambda u.u^2 = Q)(+(\lambda v.v^2, \lambda w.w^2))$

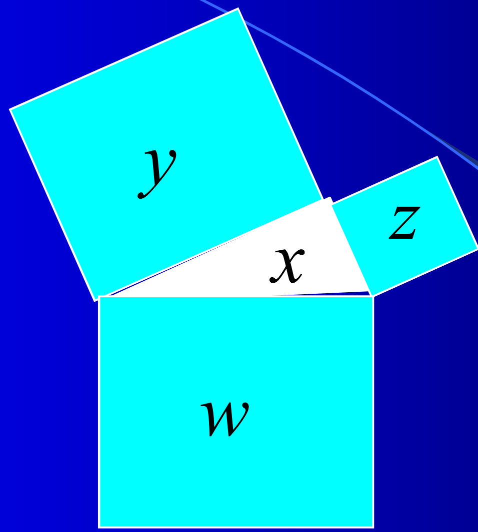
Second reinterpretation

- Reinterpretation:



- Concepts construction:
  - $\lambda x(area(w <= f_1(x)) = area(union(y, z) <= f_2(x))$
  - $\lambda u.u^2 = +(\lambda v.v^2, \lambda w.w^2)$

Program transformation rules



$$\lambda x.\lambda w.\lambda y, z.(area((w <= f_1)(x,w)) = area((union(y, z) <= f_2)(x,(y, z)))$$

$$\lambda u,v,w.u^2 = v^2 + w^2$$

Questions about diagrams

- What is their expressive power
- Why can they be interpreted so effectively
- What is the relation between logic and diagrammatic reasoning

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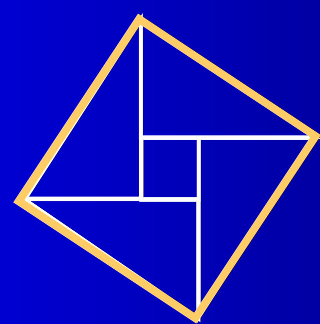
Diagrams and abstraction

- A common view is that diagrams are good for expressing concrete information but…
- There is a limitation in the abstractions that can be expressed
- The theory of graphical specificity (Stenning and Oberlander, 1995)

We can have concrete interpretations…

- Diagram:



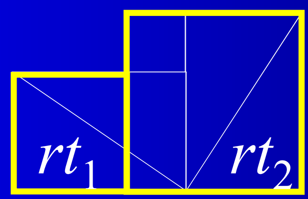
- Description:

$$sq_1 <= right\text{-}triangle(rt_1) \ \& \ square(sq_1) \ \& \ side(hipotenuse(rt_1), sq_1)$$

… and deal with the ambiguity!

We can limit the expressive power of the representational language…

Diagram:



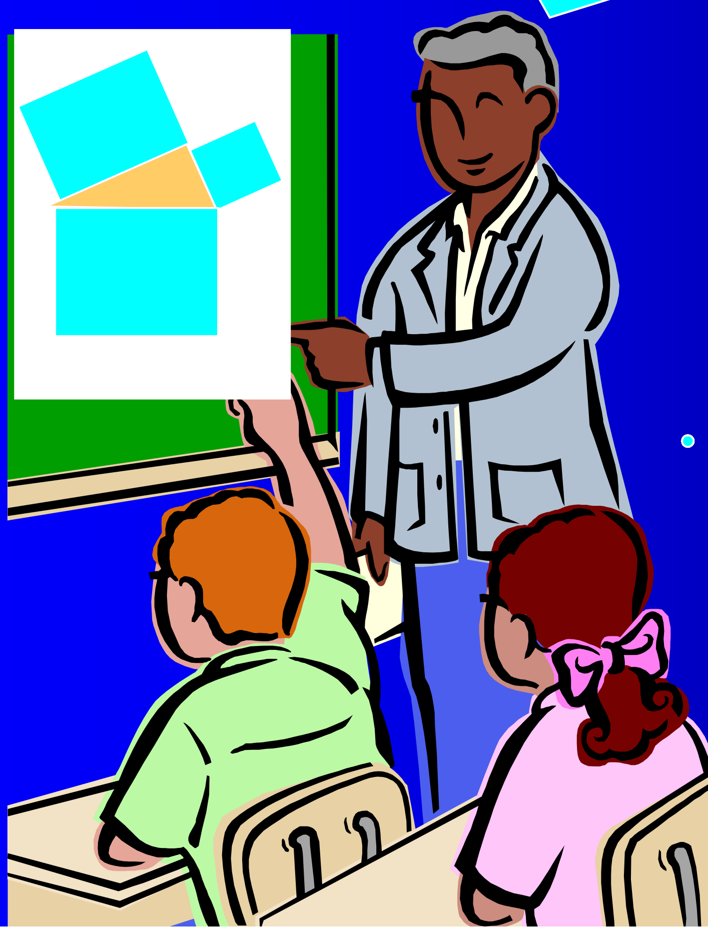
Descripción:

$$union(sq_1, sq_2) \Leftarrow right\_triang(rt_1) \;\&\; right\_triang(rt_2) \;\&\; \mathrm{square}(sq_1) \;\&\; square(sq_2) \;\&\; side(side\_a(rt_1), sq_1) \;\&\; side(side\_b(rt_2), sq_2)$$

and face the limitations of the medium!

Representation of meanings

The area on the hypotenuse of a right triangle is the same as the area of the union of the squares on its right sides



f_{TP}

Through the *lambda calculus* we represent <u>interpretations of diagrams</u> …

NOT diagrams!!!

Diagrams and abstraction

- The present theory shows that diagrams can be given generic (fully abstract) interpretations!
- A representation is specified through:
  - The external symbols and configurations
  - <u>The interpretation process</u>
  - The language to represent the interpretations <u>does not need to have a limited expressivity</u> (e.g. propositional logic)
- Diagrammatic proofs are genuine proofs!

Questions about diagrams

- What is their expressive power
- Why can they be interpreted so effectively
- What is the relation between logic and diagrammatic reasoning

Reasoning with concrete representations

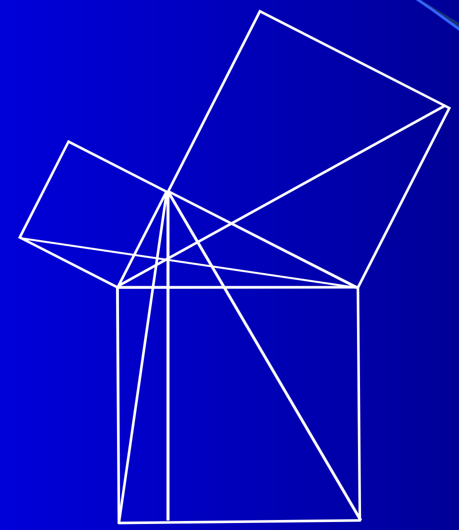
- Vision provides concrete interpretations of shapes directly
- Easy… if the problem has a concrete nature!
- Concrete problems can often be expressed through diagrams
- But, if the problem demands abstraction (e.g. an infinite number of instances) concrete resources (memory and computational time) run out very quickly!

Abstractions capture change implicitly!

- Two dimensions of change:
  - The parameters of the diagrammatic objects
  - Different diagrammatic configurations that have the same description (i.e. equivalent in relation to the task)

Abstractions account for equivalent objects!

- Diagram:

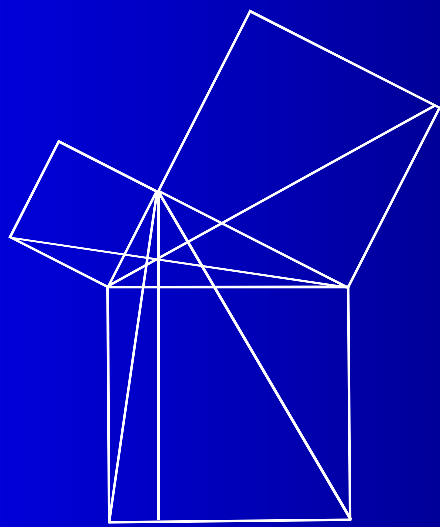


- Description: $y <= f_1$
- where:

$$f_1 = \lambda x \lambda y.right\_triang(x) \ \& \ \text{square}(y) \ \& \ side(hipotenuse(x), y)$$

Abstractions account for equivalent objects!

- Diagram:



- Description:

$union(y, z) \Leftarrow f_2$

where:

$f_2 = \lambda x \lambda y \lambda z.right\_triang(x) \;\&\; square(y) \;\&\; square(z) \;\&\; side(side\_a(x), y) \;\&\; side(side\_b(x), z)$

Diagrammatic reasoning is monotonic!

- In spite of the change in the geometric form and regardless the values of the parameters of diagrammatic objects, the synthesis of the geometric and arithmetic processes is monotonic

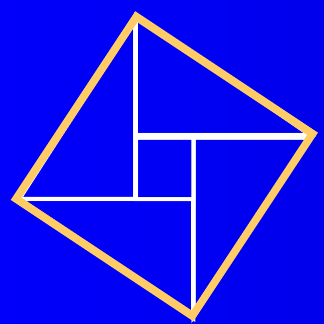
Reading a diagrammatic sequence!

- Incremental interpretation:
  - *every man is mortal*
  - $\lambda P \lambda Q \lambda x(P(x) \supset Q(x))(\mathrm{man})(\mathrm{mortal})$
  - $\lambda Q \lambda x(\mathrm{man}(x) \supset Q(x))(\mathrm{mortal})$
  - $\lambda x(\mathrm{man}(x) \supset \mathrm{mortal}(x))$
- There is not a change to account for!
- Natural language quantifiers can be seen as conservation principles!

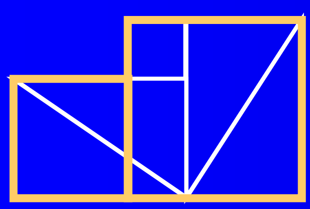
Reasoning with abstractions is easy!

- Abstractions are small finite representational objects (that represent interpretations) that can be used in thought process as units, but have a very large, perhaps infinite, extension

What is hard is to produce the relevant abstractions!

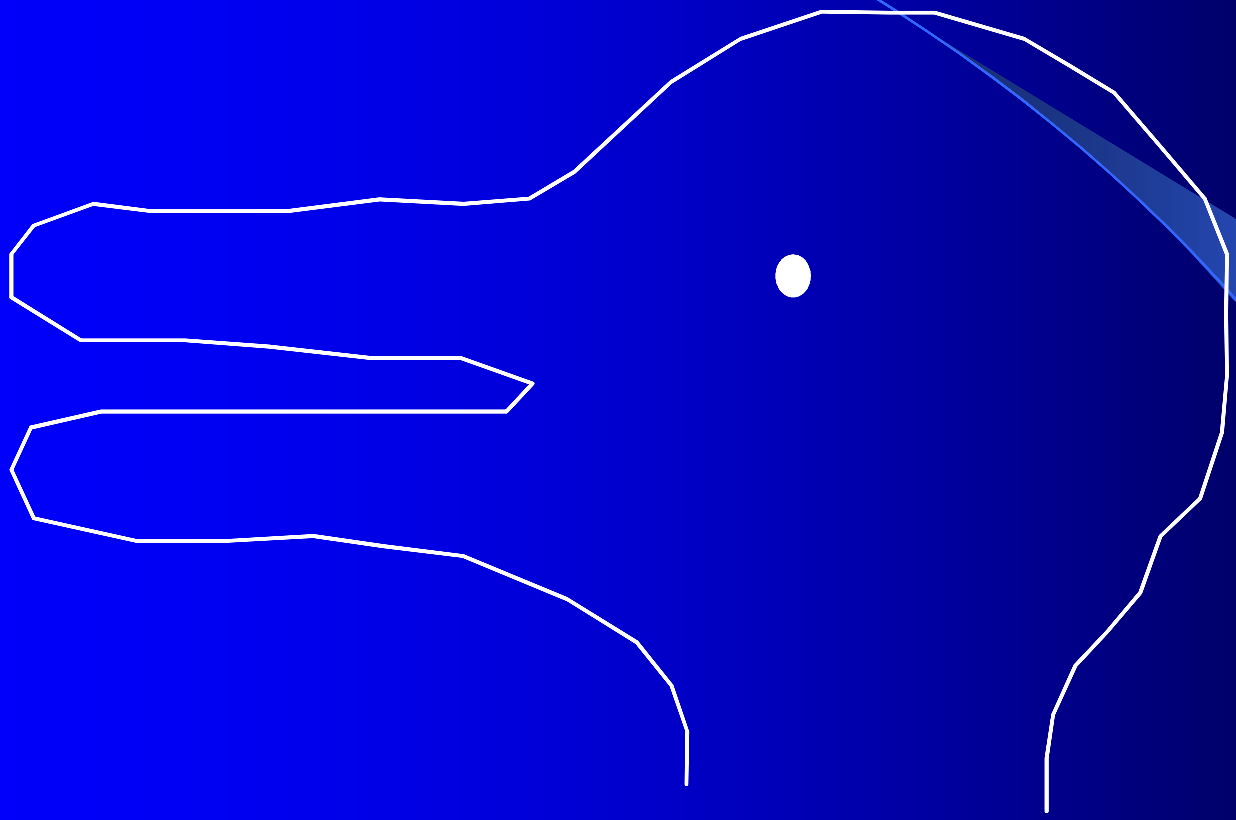


A square on the hypotenuse of a right-triangle



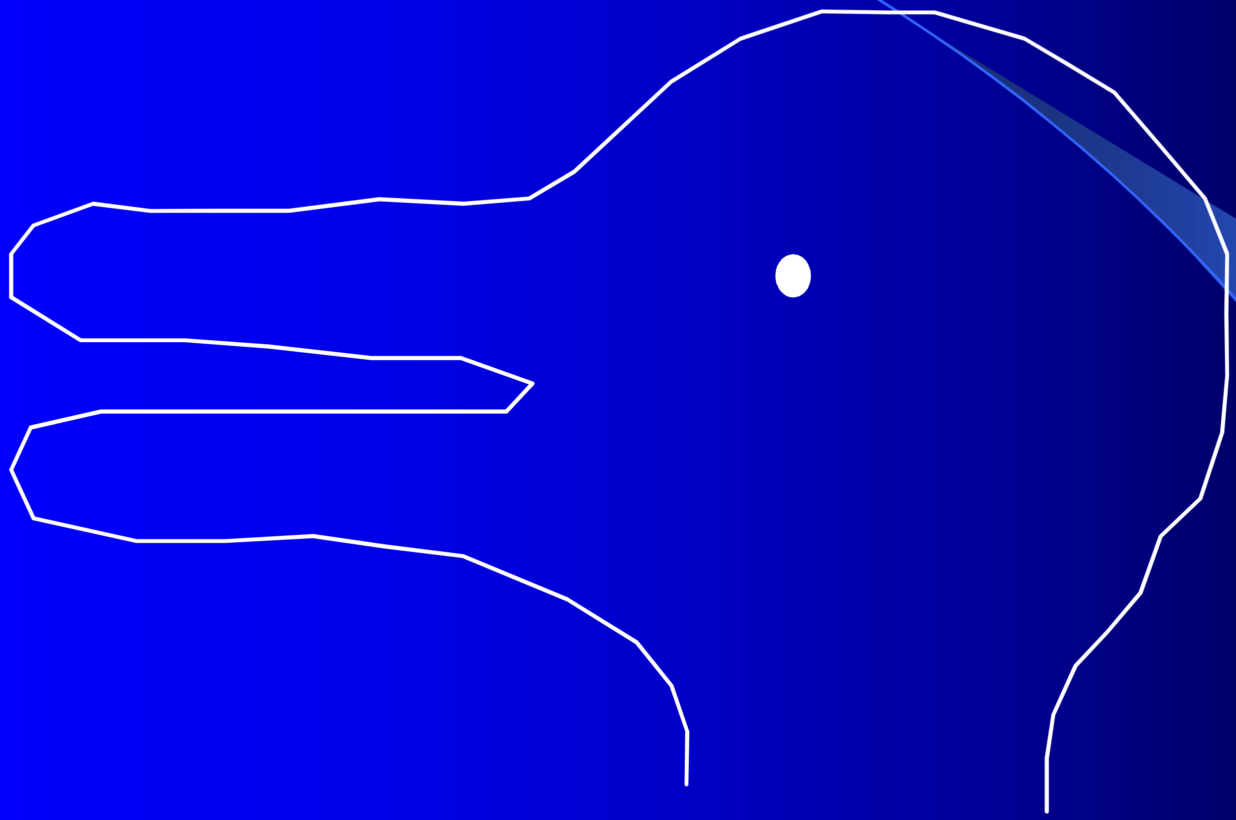
The union of a square on a right side of a right triangle and a square on the other right side of the same right triangle

Abstracting on concrete descriptions?



$duck_1 \Leftarrow duck(duck_1)$?

Constructing the abstraction directly!



$$x <= \lambda x.duck(x)$$

Generation of abstract descriptions …

- The extensional representation
- Visualisations (i.e. Reinterpretations)
- Domain knowlege (e.g. Geometry)
- Knowledge about the aims of the task (e.g. theorem proving and discovery)

Perceptual inference

Questions about diagrams

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- Why can they be interpreted so effectively
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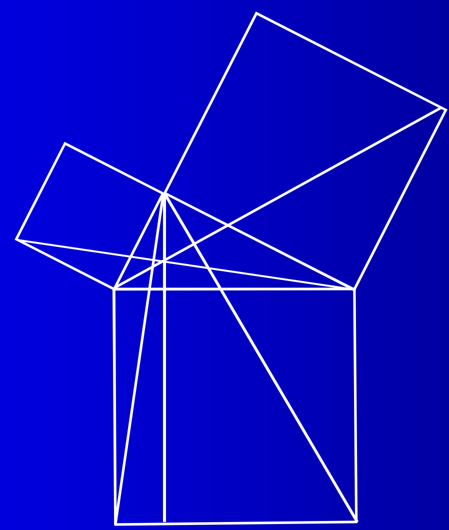
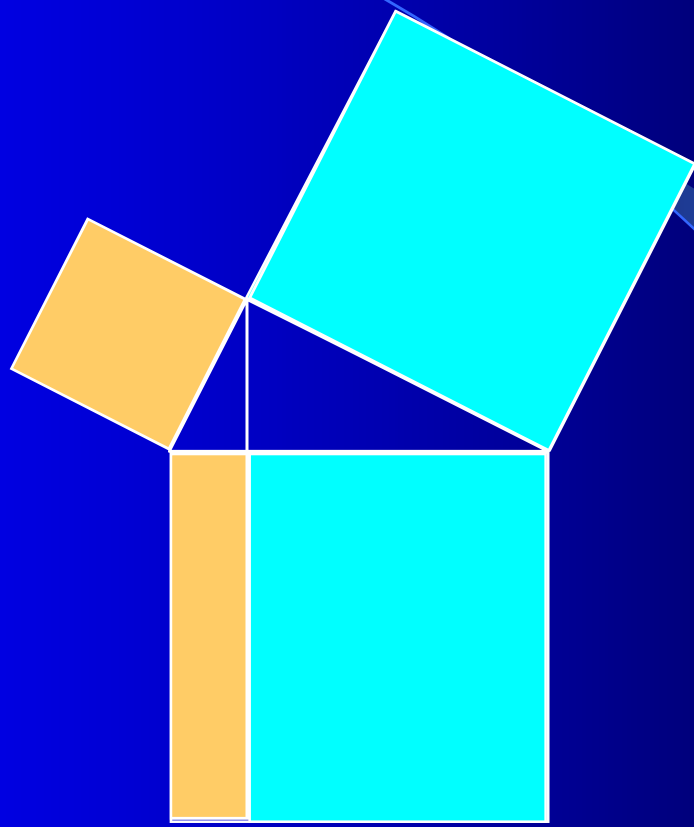
The axiomatic method



Proposition 47, Euclid's Elements

The axiomatic method



A simpler problem!

Reinterpretations

- Enrich the problem-solving space
- Interesting emerging objects belong to the enriched space
- The recognition of emerging objects depends on the interpretation process, but also on the nature of the external representation!
- The process is genuinely synthetic and synthesized objects cannot be found through analysis!

The paper:

Luis A. Pineda, Conservation principles and action schemes in the synthesis of geometric concepts, *Artificial Intelligence* 171 (March, 2007) 197-238.

Thanks very much!