Abstraction, Visualization and Graphical Proof

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Content

- Diagrammatic Proofs
- Representation of diagrams
- Abstraction and diagrams
- Pragmatics and theorem proving
- Diagrammatic theorem-proving
- Learning and graphical proof

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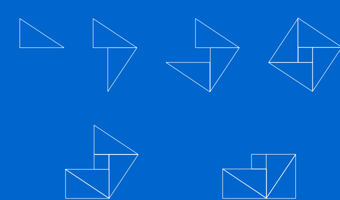
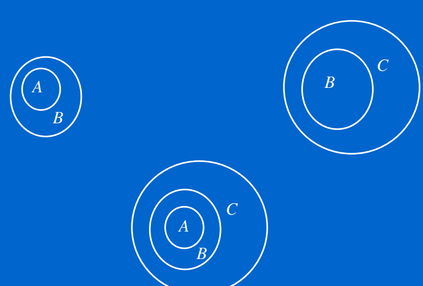
The theorem of Pythagoras

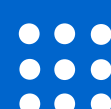


Euler Circles



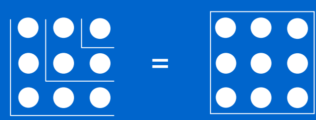
Theorem of the sum of the odds

$$1 + 3 + 5 + \cdots + (2n\text{-}1) = n^2$$



Theorem of the sum of the odds

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$



Diagrams without words?

- What kind of representations?
- Do they express the theorems?
- Do they express the proofs?
- Are these proofs valid?
- Do they help to discover the theorems?

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Representation of diagrams

- What is an analog?
- Analogical versus propositional representations
- The imagery debate
- Diagrams and Turing Machines
- Interpretation change

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Analogical representations

- Reasoning with an “analog” (Levesque and Brachman, 1985):
  - A database:

Course:

ID	Name	Dept.	Instructor
csc248	ProgLang	CompSci	Dan
mat100	HistOfMath	Maths	Luis
csc373	AI	CompSci	Ivan
…	…	…	…

Analogical representations

- A logical representation (quantifier-free, function-free atomic sentences):
  - Course(csc248), Course(mat100), Course(csc373)
  - Name(csc248, Prog. Lang.), Name(mat100, Hist. Of Maths.), ...
  - Dept(csc248, Comp. Sci.), Dept(mat100, Maths.), Dept(csc373, Comp. Sci.)
  - Instructor(csc248, Dan), Instructor(mat100, Luis), ...
- A question about the world:
  - How many courses are there in Computer Science?
- An answer about a table (In the database we count tuples):
  - Count *c* in Course where *c.Dept* = CompSci
  - 2
- But this is not the question!

Analogical representations

- For answering the question about the world we need:
  - Unique name assumption: $c_i \neq c_j$
  - Close-world assumption:
    - $\forall x[\text{course}(x) \supset x = \text{csc248} \cup ... \cup x = \text{math100}]$
  - Then, we can conclude that the individuals satisfying *course*(x) are 2!

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Analogical representations

- The analog:
  - One representational object for each individual in the domain
  - One tuple for each relationship between individuals
- Analogical reasoning:
  - Change the question: instead of asking something about the world, ask something about the structure of the representation!
- The advantage:
  - Questions can be answered directly: calculate instead of reasoning!
- The price?:
  - Nothing can be left unsaid about the domain?
  - Analogs express complete knowledge always?

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Representation of diagrams

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Propositional representations

- Pat Hayes (1974):
  - Representing Knowledge in AI (Representation Theory)
  - Scheme: Precise and systematic NOTATION
  - Configuration: a particular expression in a scheme
  - A notion of well-formedness:
    - Formal: knowledge can be used by programs
    - Informal: its interpretation requires deployment of knowledge by humans
  - Semantic theory: Model theoretic

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Propositional representations

- The syntax can vary but the semantics stays (Not to throw out the baby and keep the water!)
- A semantics allow us to say precisely what is being claimed about the world:
  - What *individuals* are there
  - What *properties* they have
  - In what *relations* they stand to each other

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Propositional representations

- Propositions: the knowledge expressed by descriptions or configurations in some scheme (with a well-defined semantics):
  - Logical representations
  - Semantic networks
  - Frame systems
  - Any formal scheme!
  - Linguistic representations?

Analogical representations

- Sloman (1971, 1975): Fregean representations (propositional) and analogical representations:
  - Both have complex structures (syntax)
  - Both have a semantics (refer, represent or denote things that have parts and relations between parts)
- The difference:
  - Analogical: representation and thing must be complex and there must be a correspondance between their structures
  - Fregean: there need be no such correspondance

Analogical representations

- If *R* is a representation of *T*:
  1. There must be parts of *R* representing parts of *T*
     - Dots and lines in a map represent cities and roads in a country
  2. It must be possible to specify a correspondance between properties and relations of parts of *R* and properties and relations of parts of *T*
     - Sizes and distances in maps represent the sizes and distances between cities in the country
  3. Relationships between parts in *T* do not need to be explicitly named in *R* (*in*, *above*, etc.).

Analogical representations

- The difference:

  (1), (2) and (3) do not hold in Fregean representations:
  - *R* = "The city 53 miles north of Brighton"
  - *T* = London
  - Brighton is not a part of London
  - London has a complex structure (the city) which bears no relation of the structure of *R*
- Fregean representation are conventional!

Analogical representations

- How the structure of a representation can correspond to the structure of a thing in the world?
  - Is it not a category mistake to compare expressions with things?
  - Do we mean spatial correspondance?
  - What about abstract objects, like numbers, theorems and proofs?
  - Do they have "spatial" structure?
  - Can they only be represented through Fregean representations?
  - What about diagrammatic proofs?

Analogical representations

- Pat Hayes again (1974):
  - Representations = Marks + Medium (a substance)!
  - Direct (analogical) representations: Direct models or pictures of the things represented
- A notion of *modality*: similary between *medium* in which *R* is embedded and *T*
  - "each medium-defined relation used in constructed configurations corresponds to a similar relation in the meanings, and the representation is a structural homomorph of the reality with respect to these relations"

Analogical representations

- Similarity between *medium* of *R* and *T*
  - A map of a room is a direct representation of the spatial relationships in the room by virtue of the 2-d plane of the paper (the medium), and the 2-d plane of floor of the room (a spatial property of a real world-entity).
  - Structural homomorph: Properties and relations that hold between symbols (marks) in the medium, hold also between the objects in the world:
    - "The paper is a direct homomorph of the room. They are the same sort of structure (2-d Euclidean space)"
  - An abstract notion of structure (for representation and thing) is required (e.g., an algebraic account: a medium as a category?)

Analogical representations

- Hayes' objections:
  - Every representation is *direct*
  - Levels of representation: "for a medium may not be physically present, but may itself be represented by configurations in some quite other medium"
  - Reduction of medium: Things represented through diagramms (e.g., a map) can also be represented by descriptions (e.g., Graflog's logical representations of diagrams, 1989, 2000)!
  - Medium: "The choice of primitive relationships defines both the medium and level at which analysis will cease"

Analogical representations

- Hayes' contention:
  - The idea that analogical representations are "more efficient" (Sloman,1971) is fallaciuos! Discussion of *efficiency* must take into account the properties of the medium.
- Sloman's pesimism:
  - "... The benefits of analogical representations can be gotten from Fregean representations, suitable organized and interpreted" ☹
  - "Thus my suggestion that AI workers interested in problem solving should design machines to solve problems looking at diagrams, maps or other spatial structures may be many years premature" ☹

Analogical representations

- Graphics are "informal representations":
  - Diagrams are *informal* because deployment of knowledge is required to express their meanings through descriptions: the level and medium where computations are really carried out!
  - "Analogic" descriptions representing diagrams are used in theorem-proving as auxiliary objects, which provide heuristic support to the real proof procedure, which is essentially logical or algebraic!

Analogical representations

- The fate of diagrams:
  - "All of the described diagrammatic reasoning systems use diagrams in search for an essentially algebraic proof of the theorem" (Jamnik, Ph. D. Thesis, pp. 35)
    - The systems:
      - Gelernter's Geometry Machine
      - Hyperproof (Barwise and Etchemendy)
      - Grover (Baker-Pummer and Bailin)
      - Other: Golstein's, Nevin's, McDougal and Polya, etc.
    - Also, Graflog (Pineda, 1989).
  - Diagrams are second kind citizens in the world of Theorem-Proving ☹

Representation of diagrams

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- Intepretation change

Are there pictures in the head?

1. Independent modules for image Generation
   - Shape information (Left hemisphere, PTO?)
     - Shape lexicon
     - Shape formation (Time varies with shape complexity)
   - Location system (Parietal cortex?)
2. Shares Inspection mechanisms (scanning) with:
   - Like-modality perception (selective interference experiments)
   - Perceptual recognition
     - Unilateral visual neglect: damage to the right parietal lobe
     - Patients with no ability to perceive shape or location, cannot do so in imagery!

Properties of imagery:

3. Retention: images occur in visual cortex (Good for vision, bad for imagery!):
   - Limited retention
   - Retention by chunks
4. Transformation in Problem-solving:
   - Rotation (time proportional to angle of rotation)
   - Scaling (time proportional to the disparity of sizes)
5. Individual differences: people good at some imagery tasks, might be weak in different tasks (independent mechanism)

Properties of imagery

In conclusion (speculative):

- Every thing points to that there are "images" in the visual cortex, with retino-topical properties, which play an important part in problem-solving
- Scanning over such "internal retina" is important
- Different "imagery modules" produce different "perceptual primitives", and different imagery problem-solving strategies!

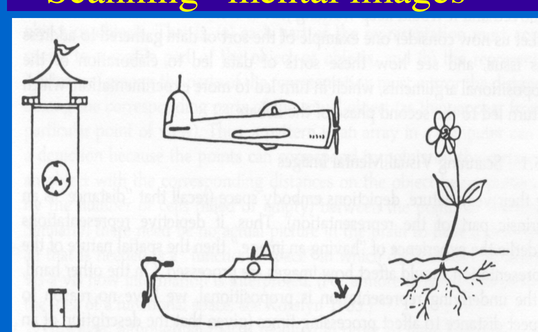
The imagery debate

- Are there pictures or spatial representations in the head?
  - The propositional account:
    - NO... It is epiphenomenal! (Pylyshyn´s)
  - The imagery position:
    - Imagery parallels perception (Kosslyn, Farah)
    - Visual (Modality specific)
    - Spatial (Modality independent)

Scanning "mental images"

- Imagery position: Scanning time is a proof that there are images!
- Propositional position: Scanning reflects a property of a propositional representational format (e.g., a semantic network!)

Scanning "mental images"



Kosslyn's imagery test

1. Subjects memorize the drawings
2. Subjects close their eyes
3. Subjects listen to the name of an object
4. Subjects visualize it: stare at it with the "mind's eye" focusing at one end of the image.
5. The name of an object´s component is presented on tape. Half the times the part is in the object; the other half, it isn't.

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Kosslyn's imagery test

6. Subjects look for the named component on the image object; parts were at the end or in the middle of the object.
7. Subject press the *true button* when the "see" the object, or the *false button* if the component is not there

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Kosslyn's imagery test

- The hypothesis:
  – If images representation depict information directly, it ought to take more time to locate parts farther from the point of focus (the scanning time)

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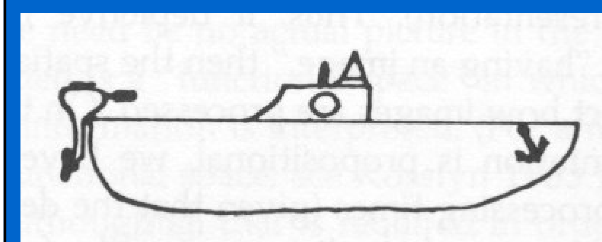
Scanning "mental images"



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How long does it take?

Motor to Porthole

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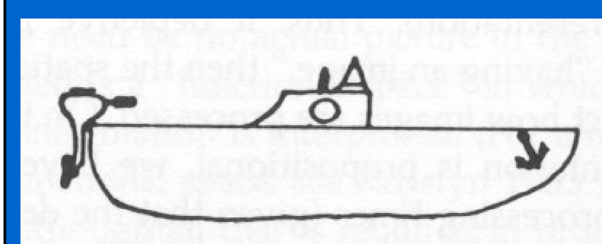
Scanning "mental images"



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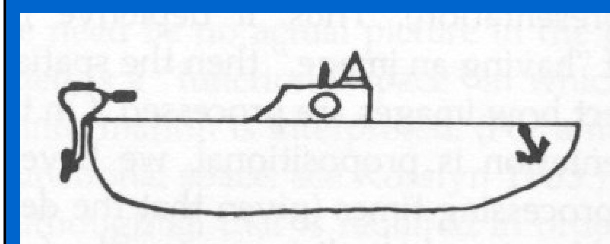
How long does it take?

Motor to Anchor

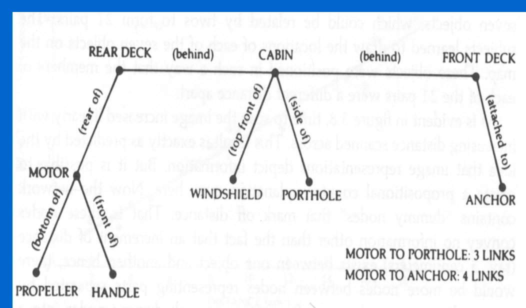
The propositional answer!

- Bobrow (cited by Kosslyn):
- A propositional representation for the boat
  - BOTTOM-OF(PROPELER, MOTOR)
  - REAR-OF(MOTOR, REAR-DECK)
  - BEHIND(REAR-DECK, CABIN)
  - BEHIND(CABIN, FRONT-DECK)
  - …

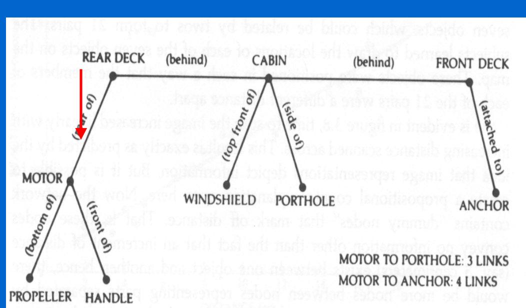
Scanning "mental images"



The format: a semantic network



Motor to Porthole



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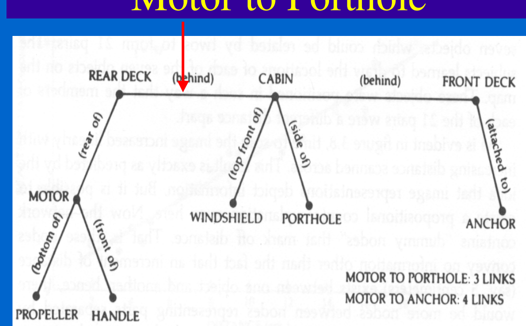
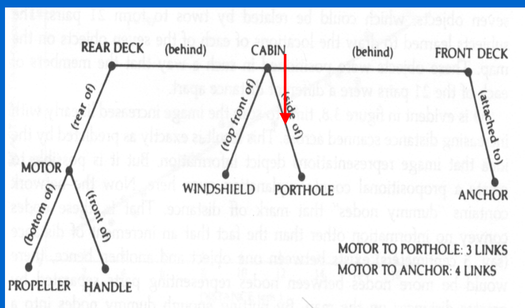
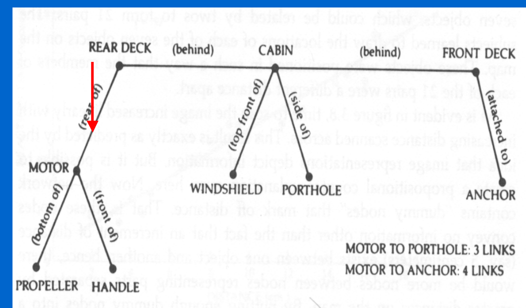


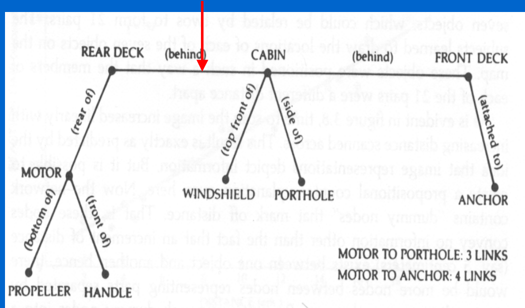
Motor to Porthole: 3 time units



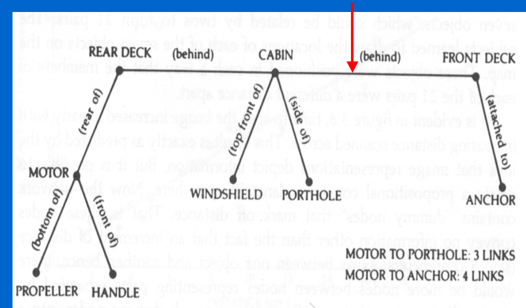
Motor to Anchor



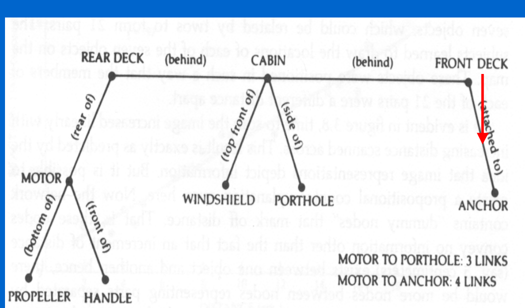
Motor to Anchor



Motor to Anchor



Motor to Anchor: 4 time units!



Kosslyn's imagery test

- The hypothesis:
  - If images representation depict information directly, it ought to take more time to locate parts farther from the point of focus (the scanning time)
- The result:
  - It did happen (It is a lineal relation)!
- The conclusion:
  - There are images and a scanning device!
  - Perceptual primitives?

Representation of diagrams

- Whats an analog?
- Analogical versus propositional representations
- The imaginery debate
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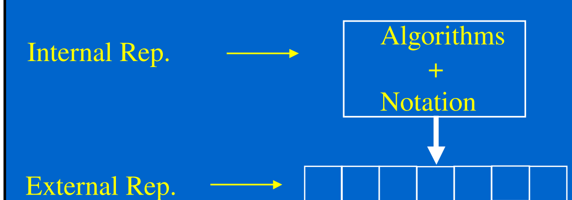
- Interpretation change

Linear Turing Machines

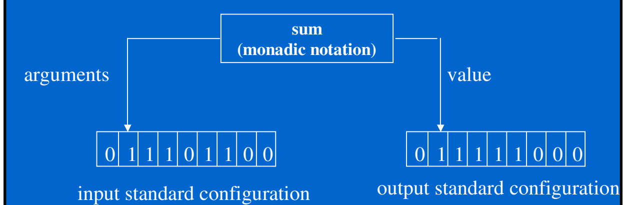
- Properties of Turing machines:
  - Alphabet, states, transition tables
  - Notation: implicit in the algorithm
  - What is the interpretation of "111"
    - three? (an analogical representation?)
    - seven?
    - one hundred and eleven?

Where is the notation?

- External versus internal representations:

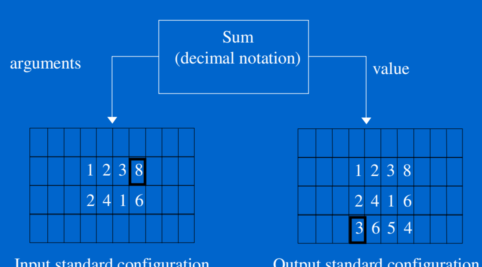


Standard interpretation conventions



All Turing machine computes a function!

Bi-dimensional Turing Machine



Computacional effectiveness

- Computation effectiveness
  - The same function but different algorithms
- Lineal tape and monadic notation
  - Linear complexity
- Bi-dimensional tape and decimal notation
  - Logarithmic complexity
- Limited computational resources

Local representations

- Turing machines:
  - Local or discrete cells
  - local or discrete symbols
  - Discrete transitions
  - Local or discrete scanner device

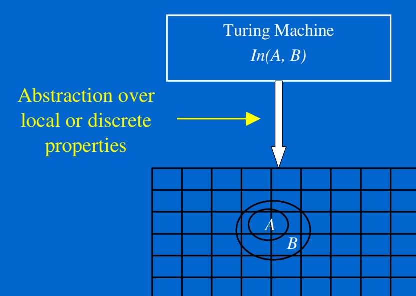
TM and Prop. Representations

- A Semantic Network:
  - The interpreter:
    - Knowledge of an algorithm and notation
    - An internal representation
  - The external representation:
    - A local scanning device with discrete transitions
    - A medium has the shape of a graph!
- Sloman explicated!
  - The shape of tapes is analogous to the structure of the object!
  - TM's TAPEs need to have a shape!
  - There is no computation without a medium!

Local versus global representations

- Diagrammatic Turing Machines:
  - Local or discrete cells?
  - Discrete symbols (not necessarily local to cells!)
  - Discrete transitions
  - But… a general or global scanner device: The eye!

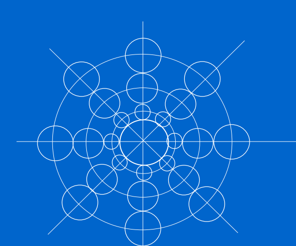
Generalized scanner device



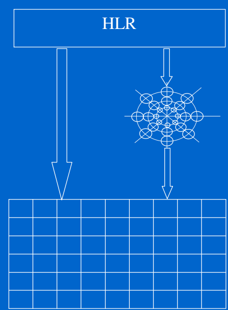
Diagrammatic Turing Machine

- Turing Machine with:
  - Bi-dimensional Tape
  - Global scanning device
- An example:
  - A computational retina: Whisper (Funt, 1980)

Whisper's retina (Funt, 1980)



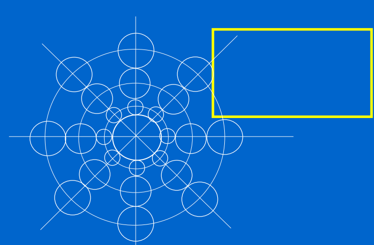
Whisper's architecture



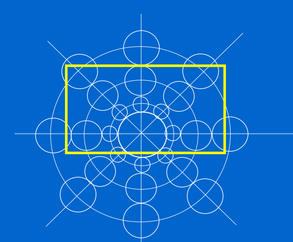
Whisper's perceptual operation

- Fixate retina on a point
- Identify a body
- Finding its center of area
- Rotate a body
- Detect the collision point between two bodies
- Detecting the center of area of a body to the retina's center

Fixate retina

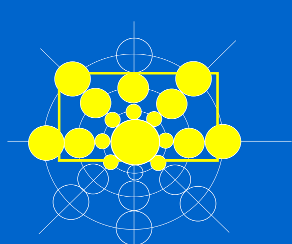


Fixate retina



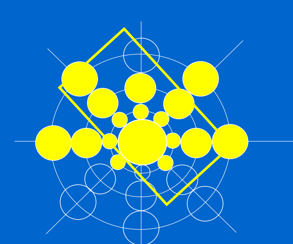
Object on the visual field

Visualize object

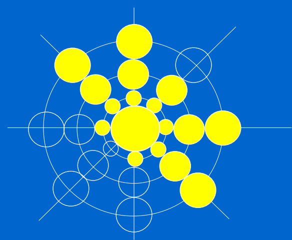


Retinal perception

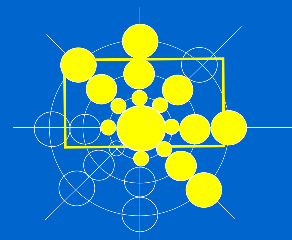
Visualization of a rotation



Visualization of a rotation

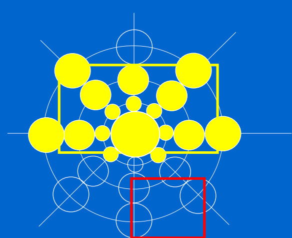


Visualization of a rotation



The object is not altered!

Collision detection



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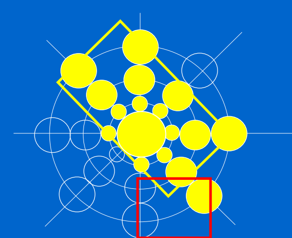

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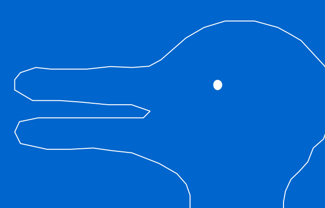
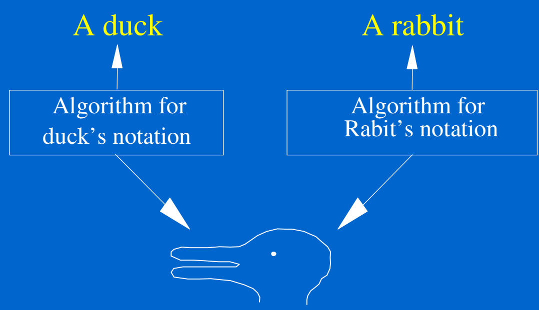
Reinterpretation of pictures



Wittgenstein: Philosophical Investigations

Computational re-interpretation



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Abstraction and FOL

- Abstraction and Mathematical logic
  - Infinite domains
  - *Complete* knowledge
  - Quantifiers range over numbers and sets
  - Conditionals permit to state what properties they have
- Independent of the representational format!

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Abstraction and FOL

- Abstractin and KR in AI: FOL permits to deal with *incomplete* knowledge:
  - $\neg p$ states what p is not without saying what it is!
  - $p \cup q$ states that either p or q, but it does not says which!
  - $\exists x\ p(x)$ states that there is a least one individual, but it does not say who is it!
  - $\forall x\ p(x)$ states that every one has p, but it does not say who they are, or whether there are any!
- Relative to a representational format!

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Abstraction and FOL

- What kind of abstraction do we need?
  - Graphical proofs are meant to be full abstractions!
  - Graphical proofs have a concrete representations, as an essential feature!
- To be or not to be!
- How the abstraction can be launch from the platform of *the concrete*?

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The KR Trade-off

- There is a trade-off between the expressive power of a representational language and its computational tractability (Levesque and Brachaman, 1985)
- *Vivid* representations: very limited expressive power (no abstraction) but tractable
- *Analogs* are *vivid* representations?

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Graphical Specificity

- Theory about the effectiveness of graphical representations (Stenning and Oberlander, 1994)
- Taxonomy of representational systems:
  - Minimally abstract (MARS)
  - Limited abstraction (LARS)
  - Unlimited abstraction (UARS)

Graphical Specificity

- Interpretation conventions
  - Terminological keys (KL-ONE T-BOX)
  - Assertional keys (KL-ONE A-BOX)
- Syntactic reflex: depends on the limitations that the system imposes on the perceptual and abstraction process.

Minimal abstraction (MARS)

- Only terminological keys
- Each symbol has a univocally determined interpretation
- Complete knowledge (configurations denote fully determined state of affairs)
- Configurations are satisfied by only one model (in the model theoretical sense)
- Examples:
  - Arithmetic operations
  - Comprehensive sentences of the propositional calculus

A comprehensive sentence

	a	b	c	d
P	1	1	0	0
Q	0	1	1	1
R	0	1	0	1

$$P(a) \wedge P(b) \wedge \neg P(c) \wedge \neg P(d) \wedge \neg Q(a) \wedge Q(b) \wedge Q(c) \wedge Q(d) \wedge \neg R(a) \wedge R(b) \wedge \neg R(c) \wedge R(d)$$

A comprehensive sentence

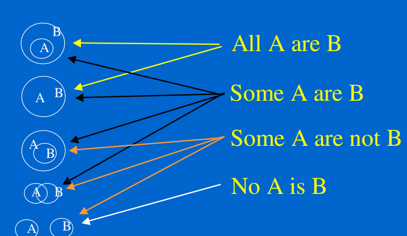
	a	b	c	d
P	1	1	0	0
Q	0	1	1	1
R	0	1	0	1

$$P(a) \wedge P(b) \wedge \neg P(c) \wedge \neg P(d) \wedge \neg Q(a) \wedge Q(b) \wedge Q(c) \wedge Q(d) \wedge \neg R(a) \wedge R(b) \wedge \neg R(c) \wedge R(d)$$

Equivalent representations but different syntactic reflex

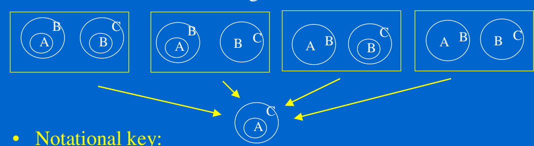
Euler Circles as MARS

Gergonne relations



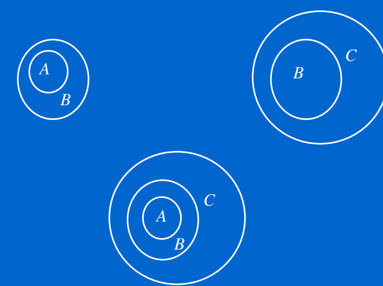
Minimal abstraction (MARS)

- Syllogistic reasoning:
  - All A are B (two diagrams)
  - All B are C (two diagrams)
  - All A are C
- We have to consider 4 diagrams!



- Notational key:
  - Each region represents a type (complete knowledge)

But we just have to consider one!



Limited abstraction (LARS)

- Only terminological keys
- The interpretation of all symbols does not need to be fully determined
- Configurations express partial knowledge
- A configuration can be satisfied by more than one model :
  - Variables in arithmetic
  - Incomplete sentences of the propositional calculus

A non-comprehensive sentence

	a	b	c	d
P	1		0	0
Q	0	1		1
R		1	0	1

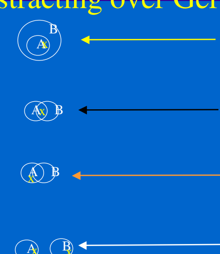
$$P(a) \wedge \neg P(c) \wedge \neg P(d) \wedge \neg Q(a) \wedge Q(b) \wedge Q(d) \wedge R(b) \wedge \neg R(c) \wedge R(d)$$

A non-comprehensive sentence

	a	b	c	d
P	1		0	0
Q	0	1		1
R		1	0	1

$$P(a) \wedge (P(b) \vee \neg P(b)) \wedge \neg P(c) \wedge \neg P(d) \wedge \neg Q(a) \wedge Q(b) \wedge (Q(c) \vee \neg Q(c)) \wedge Q(d) \wedge (R(a) \vee \neg R(a)) \wedge R(b) \wedge \neg R(c) \wedge R(d)$$

Characteristic diagrams (abstracting over Gergonne relations)



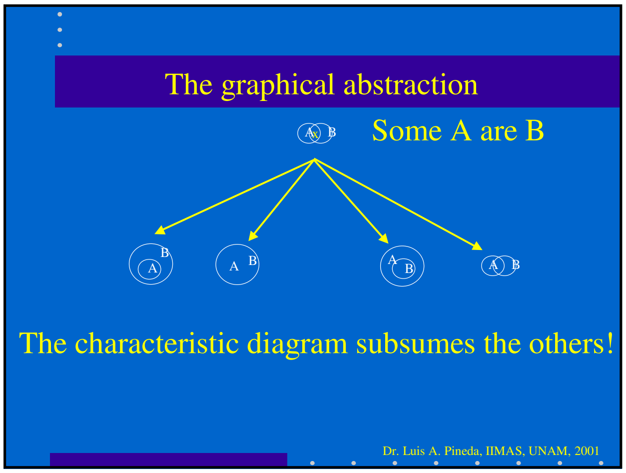
All A are B

Some A are B

Some A are not B

No A is B

x: Types of individuals implied by the premises
Types without x can or cannot contain individuals


The graphical abstraction
Some A are B
The characteristic diagram subsumes the others!
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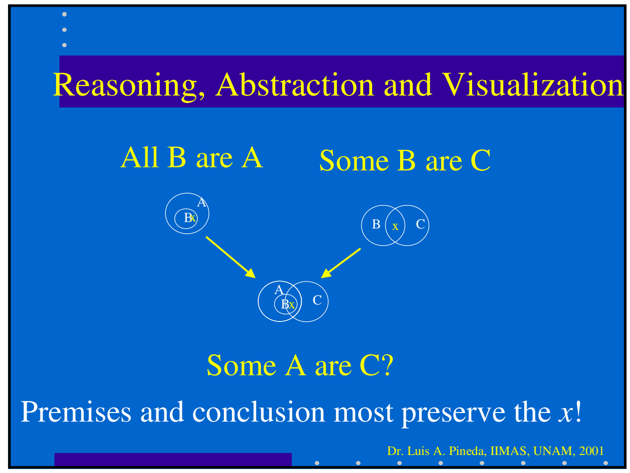
Reasoning, Abstraction and Visualization
All B are A
Some B are C
Some A are C?
Premises and conclusion most preserve the *x*!
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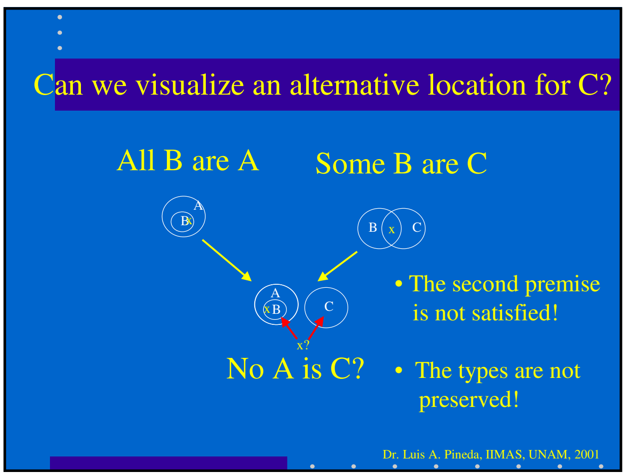
Can we visualize an alternative location for C?
All B are A
Some B are C
No A is C?
• The second premise is not satisfied!
• The types are not preserved!
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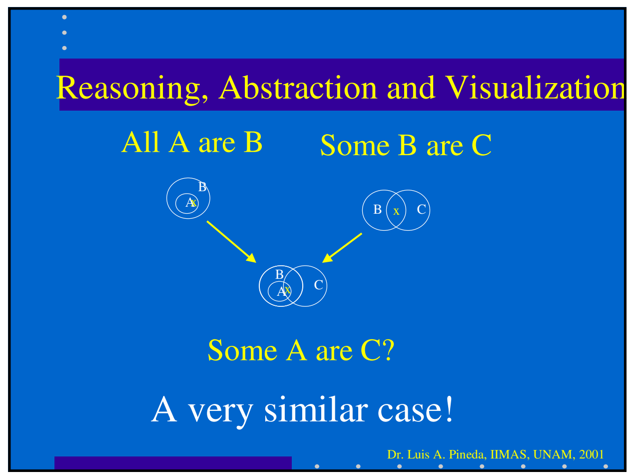
Reasoning, Abstraction and Visualization
All A are B
Some B are C
Some A are C?
A very similar case!
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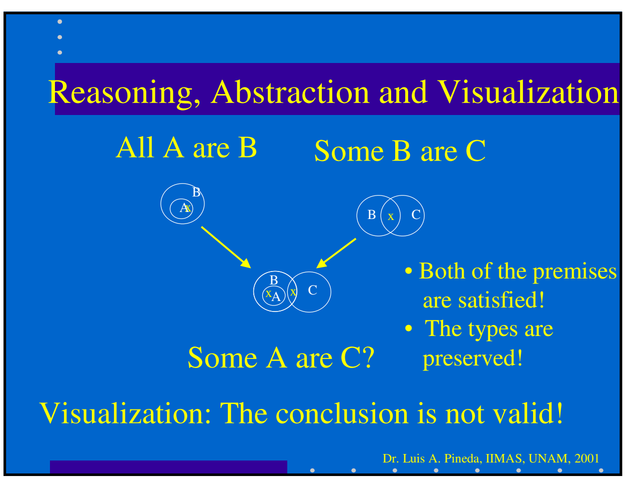
Reasoning, Abstraction and Visualization
All A are B
Some B are C
Some A are C?
• Both of the premises are satisfied!
• The types are preserved!
Visualization: The conclusion is not valid!
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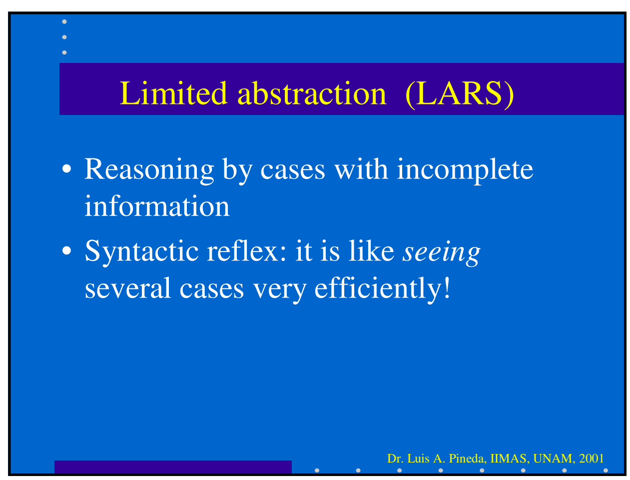
Limited abstraction (LARS)
• Reasoning by cases with incomplete information
• Syntactic reflex: it is like *seeing* several cases very efficiently!
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Unlimited abstraction (UARS)

- Assertional keys: General statements
  - First order logic
  - Natural languaje
- Graphical UARS?

Unlimited abstraction (UARS)

- Assertional keys (Stenning and Oberlander):
  - Terminological key: write “1” or “0” en d3
  - Assertional key: write “1” in d3 if c3 = e8 and “0” otherwise!

	1	2	3	4	5	6	7	8
a								
b								
c			0					
d			1					
e								0

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Limitations of S & O

- A notation states how to interpret expressions of a representation scheme: it is a question of *meaning*!
- Facts are asserted through expressions of the representational system!
- Universal statements about notation are not facts about the world
- Universal statements about notation introduce abstraction, not necessarily limited!

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Notational Keys

General statements about notation

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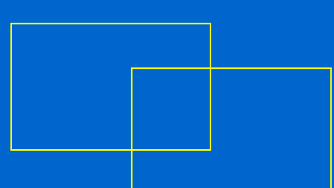
Graflog (Pineda, 89)

> These are subjects



Terminological key (graphical predicates)

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Graflog (Pineda, 89)

> This is linguistics

Terminological key (graphical constants)

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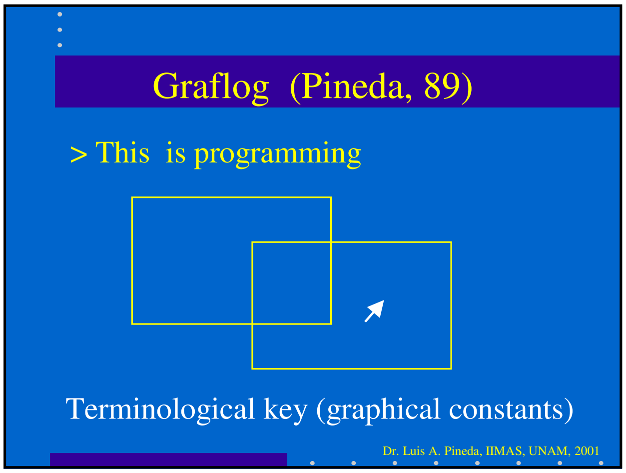
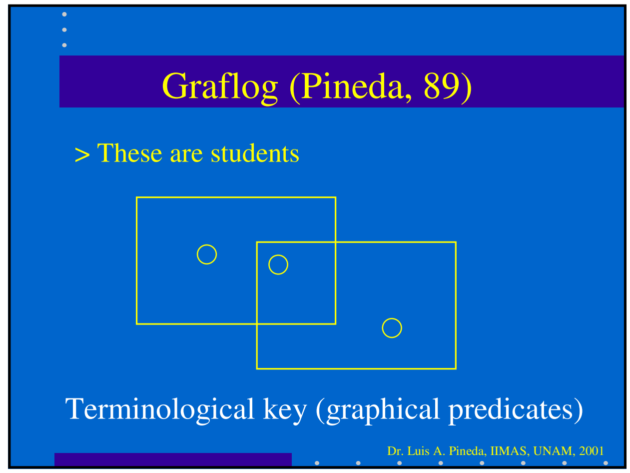
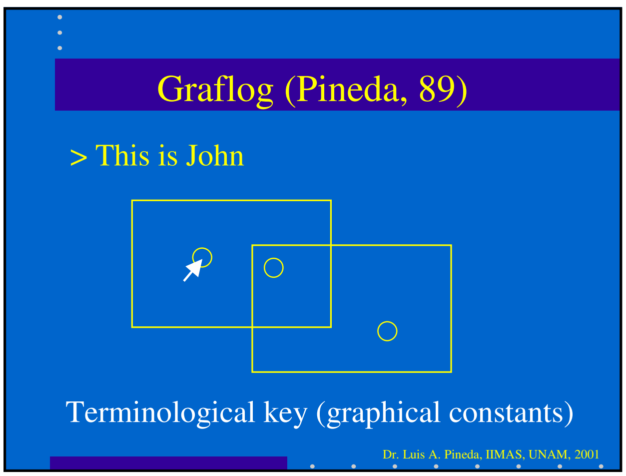
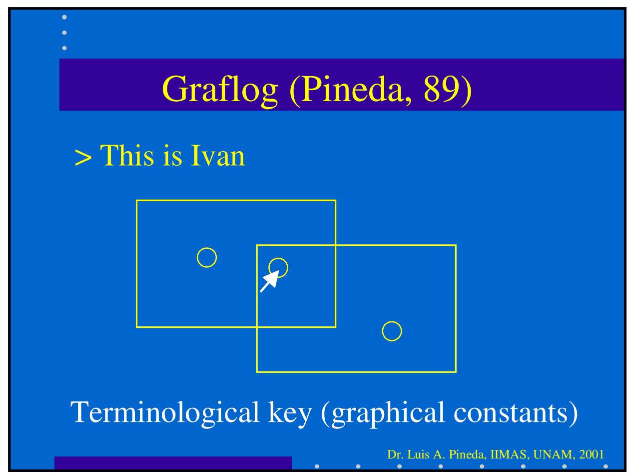
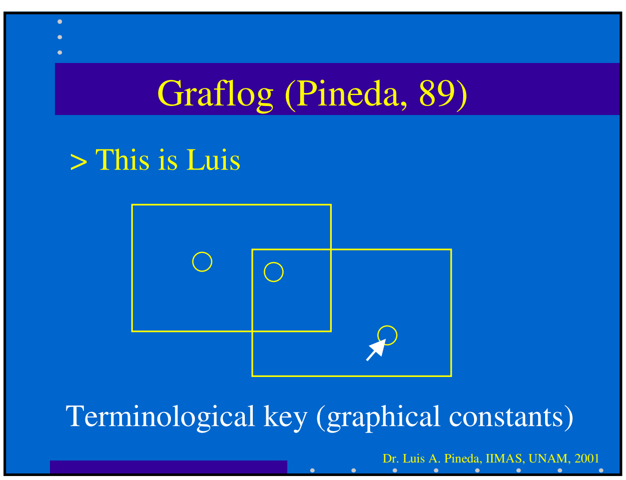

Graflog (Pineda, 89)
> This is programming
Terminological key (graphical constants)
Dr. Luis A. Pineda, IIMAS, UNAM, 2001


Graflog (Pineda, 89)
> These are students
Terminological key (graphical predicates)
Dr. Luis A. Pineda, IIMAS, UNAM, 2001


Graflog (Pineda, 89)
> This is John
Terminological key (graphical constants)
Dr. Luis A. Pineda, IIMAS, UNAM, 2001


Graflog (Pineda, 89)
> This is Ivan
Terminological key (graphical constants)
Dr. Luis A. Pineda, IIMAS, UNAM, 2001


Graflog (Pineda, 89)
> This is Luis
Terminological key (graphical constants)
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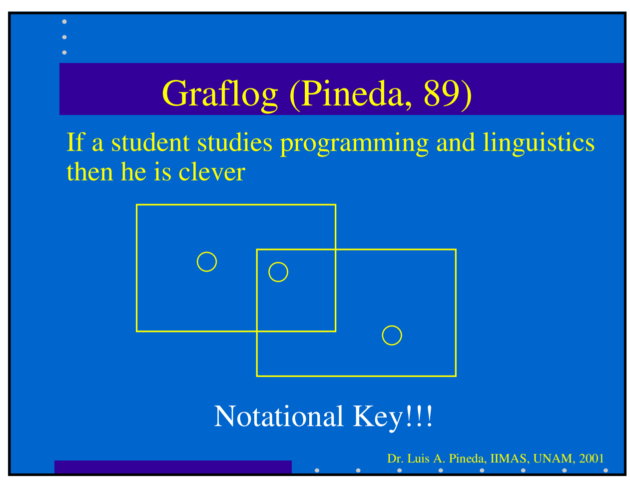
Graflog (Pineda, 89)
If a student studies programming and linguistics then he is clever
Notational Key!!!
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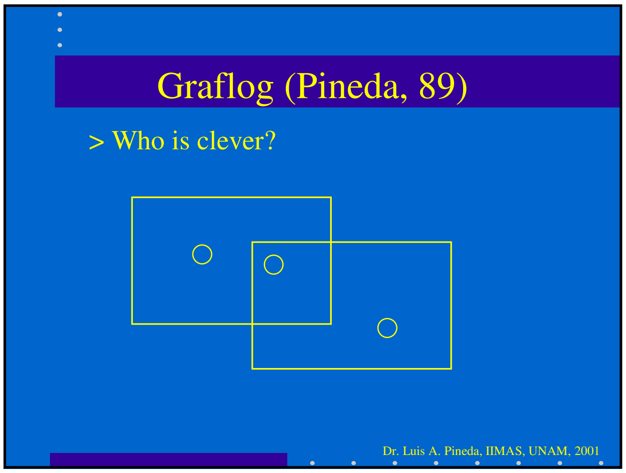
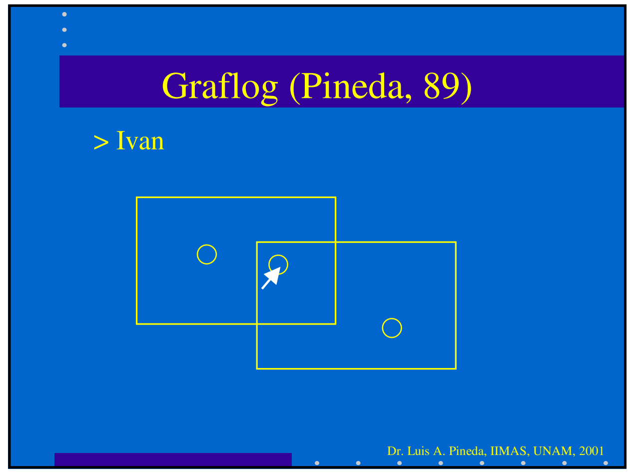
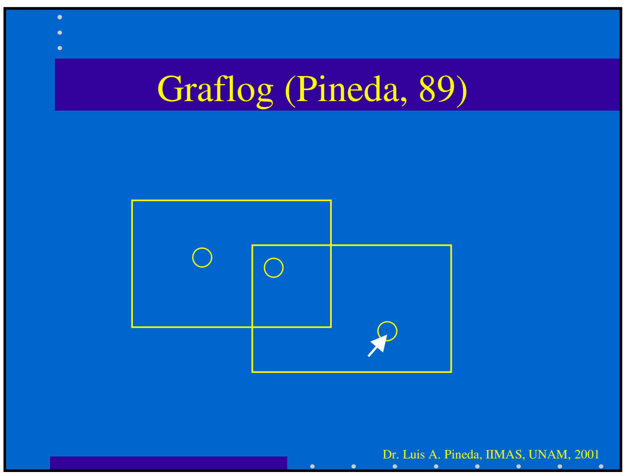
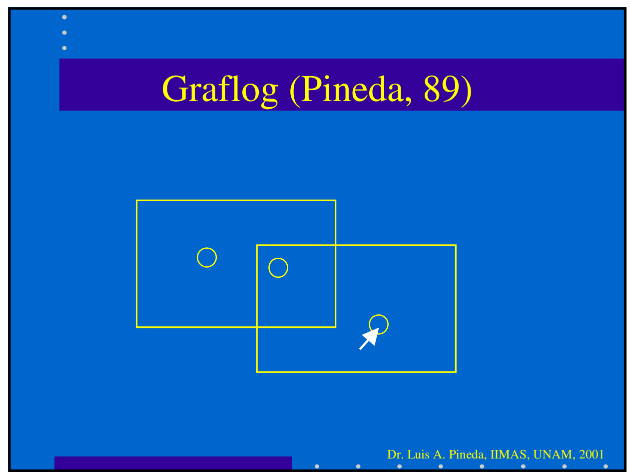
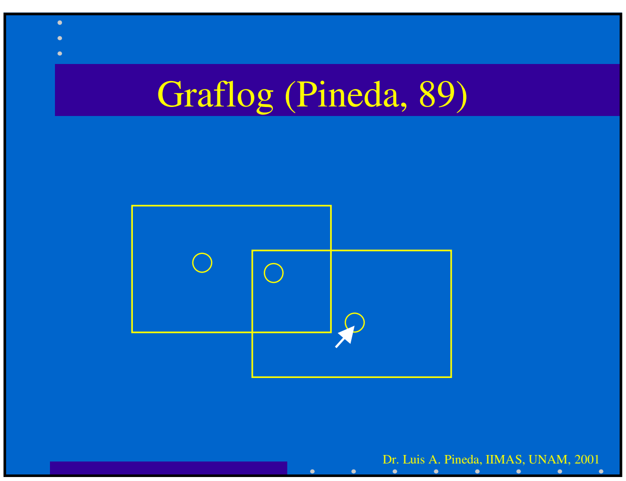

Graflog (Pineda, 89)
> Who is clever?
Dr. Luis A. Pineda, IIMAS, UNAM, 2001


Graflog (Pineda, 89)
> Ivan
Dr. Luis A. Pineda, IIMAS, UNAM, 2001


Graflog (Pineda, 89)
Dr. Luis A. Pineda, IIMAS, UNAM, 2001


Graflog (Pineda, 89)
Dr. Luis A. Pineda, IIMAS, UNAM, 2001


Graflog (Pineda, 89)
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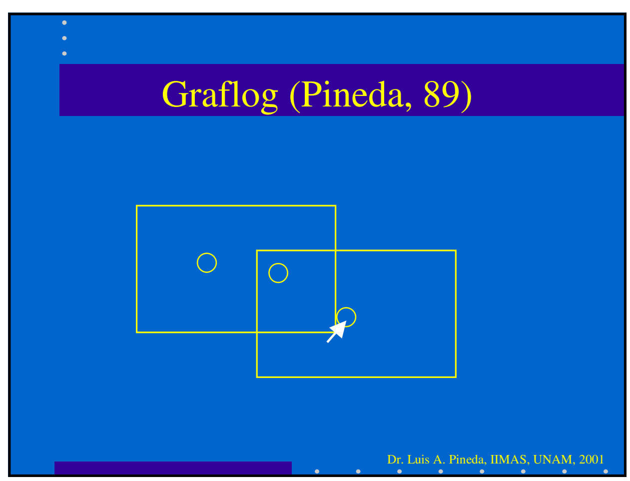
Graflog (Pineda, 89)
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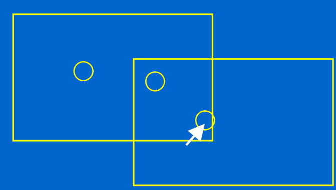
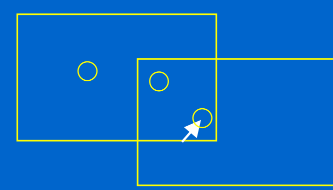
Graflog (Pineda, 89)



Graflog (Pineda, 89)



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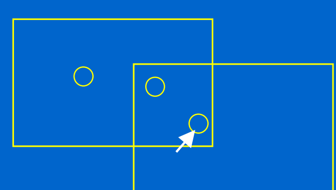
Graflog (Pineda, 89)

> Who is clever?



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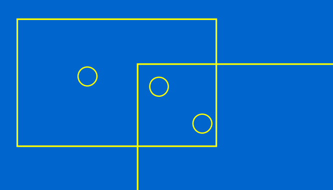
Graflog (Pineda, 89)

> Ivan and Luis!



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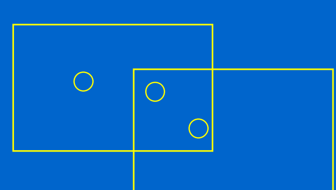
Graflog (Pineda, 89)

> Is John clever?



Incomplete knowledge!

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Abstraction in Graphics?

- Notational keys introduce abstraction:
  - General statements
  - Incomplete knowledge
- Representational keys again:
  - Terminological keys ⇒ interpretation of *atomic symbols* of different categories
  - Assertional keys ⇒ Facts expressed through expressions
  - Notational keys ⇒ General statements about notation

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Abstraction in Graphics?

- Limited abstraction?
  - Finite and often small interpretation domain!
  - Concrete interpretation: provide a model for a theory!
  - Model based Reasoning?
- For diagrammatic proofs and theorems we need full abstraction expressed through graphics!

Hierarchy of Truth

- Logical (Tautologies)
  - All possible models (independent of model)
- Synthetic (mathematical truths)
  - True in relation to every mathematical model
- Scientific (empirical evidence)
  - True in relation to the physical world
- Contingent
  - True in some models

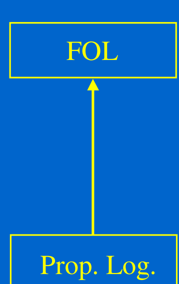
Augmenting S & O

- The theory of the specificity of graphics needs to distinguish LARS from UARS formally!
- The fact that UARS have assertional keys won't do!

Principle of Compositionality

- The meaning of a composite expression is a function of the meaning of its parts, and the mode of grammatical composition
- Terminological keys for interpreting constant symbols (of different syntactic types)
- Rules of interpretation for interpreting composite expressions
- Only terminological keys are involved!

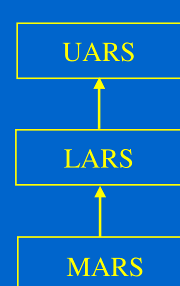
"Standard" hierarchy of abstraction!



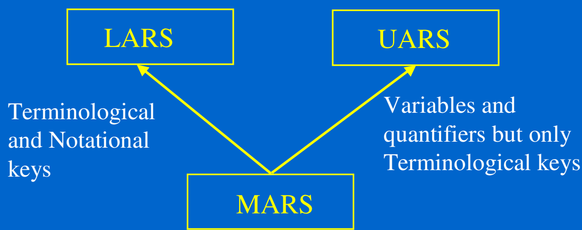
Variables and quantifiers but only terminological keys

Constant symbols and terminological keys

S&O hierarchy of abstraction!

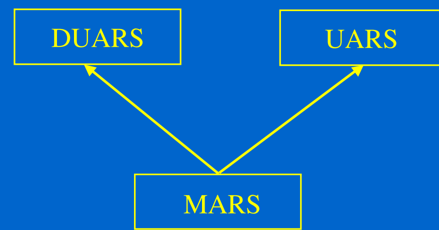


A new hierarchy of abstraction!



Global scanning: for facts *but also for NOTATION!*

Diagrammatic Unrestricted-Abstraction



Holistic versus local interpretation!

The KR Trade-off revisited!

- There is a trade-off between the expressive power of a representational language and its computational tractability relative to the kinds of representational keys involved!
- Syntactic reflex: A property of the kind of scanning device!
- Graphics: a better trade-off between the expression of abstraction and tractability

Content

- Diagrammatic Proofs
- Representation of diagrams
- Abstraction and diagrams
- Pragmatics and theorem proving
- Diagrammatic theorem-proving
- Learning and graphical proof

Pragmatics of theorem-proving

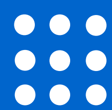
- Reasoning *with* versus *from or about* the system:
  - Selection of representational media
  - Selection of notation

Pragmatics of theorem-proving

- Inferential cost:
  - Syntactic reflex
  - The cost of the ticket (i.e., Euler Circles versus linguistic syllogisms)
- Systematicity:
  - Free and cheap tickets can have unexpected destinations

Systematicity

- The theorem:



$1 + 3 + 5 + \ldots + (2n\text{-}1) = n^2$

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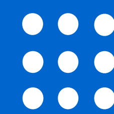
Systematicity:

- Alternative interpretations:



$1 + 2 + 3 + \ldots + (n\text{-}1) + n + (n\text{-}1) + \ldots + 2 + 1 = n^2$

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Systematicity

- There must be restrictions of syntactic and semantic mappings
- How these restrictions should be placed?

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Systematicity

- A representational key defines how to interpret a diagram in a particular representational setting
- A diagrammatic proof exists if and only if the diagram, the interpretation and the theorem are related in an appropriate manner!

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Content

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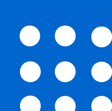
Graphical Theorems and Proofs?

$1 + 3 + 5 + \cdots + (2n\text{-}1) = n^2$



Does the diagram express the theorem?
Does the diagram express the proof?

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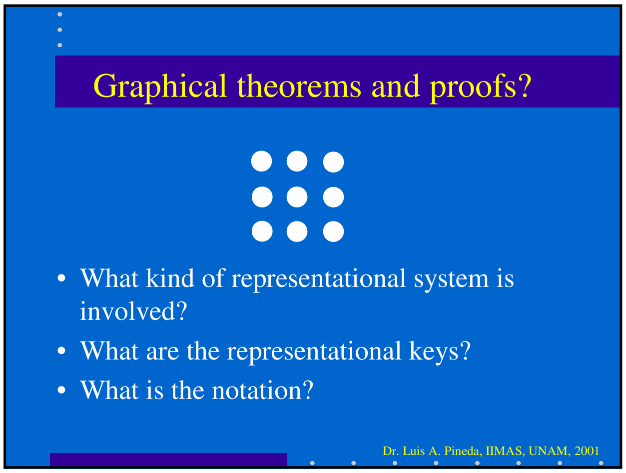
Graphical theorems and proofs?
• What kind of representational system is involved?
• What are the representational keys?
• What is the notation?
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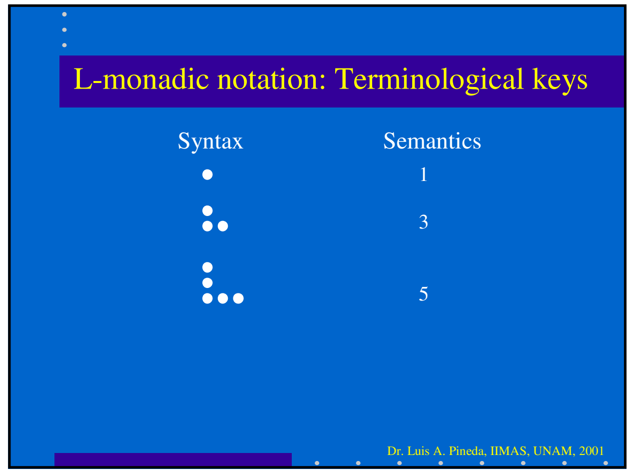
L-monadic notation: Terminological keys
Syntax
Semantics
1
3
5
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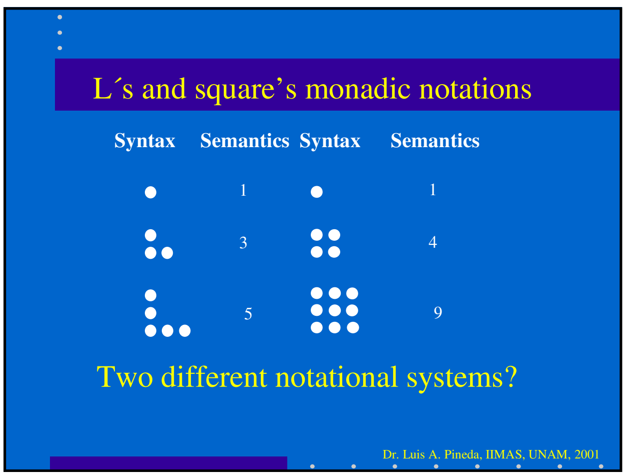
L´s and square's monadic notations
Syntax
Semantics
Syntax
Semantics
1
1
3
4
5
9
Two different notational systems?
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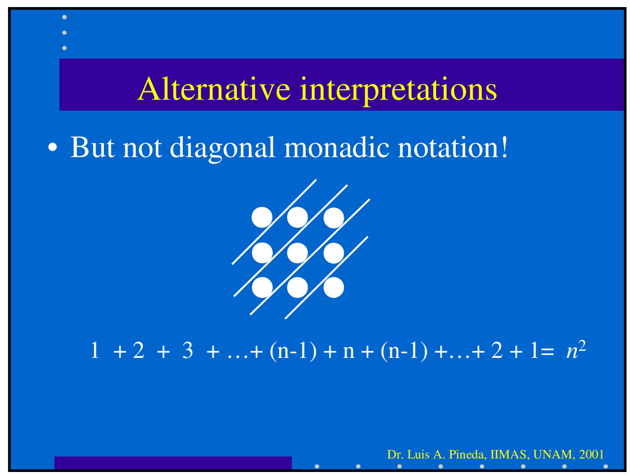
Alternative interpretations
• But not diagonal monadic notation!
1 + 2 + 3 + …+ (n-1) + n + (n-1) +…+ 2 + 1= n2
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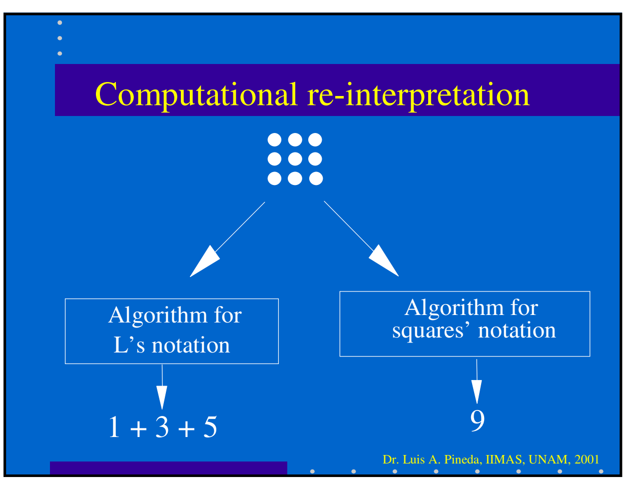
Computational re-interpretation
Algorithm for L's notation
Algorithm for squares' notation
1 + 3 + 5
9
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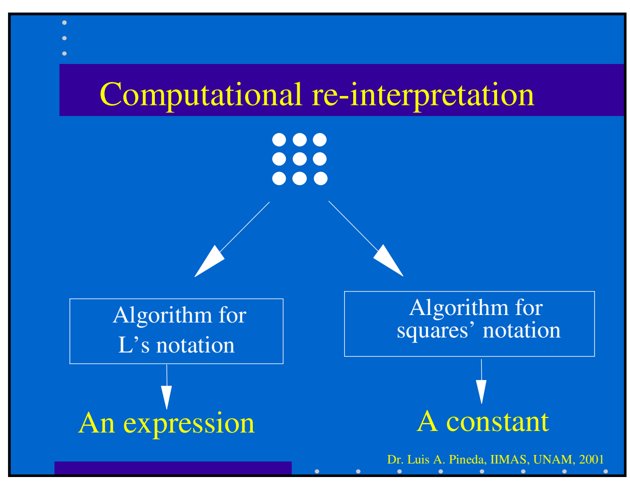
Computational re-interpretation
Algorithm for L's notation
Algorithm for squares' notation
An expression
A constant
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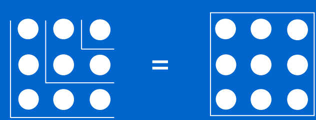
Interpretation Change!

Reasoning not within but *about* the system:



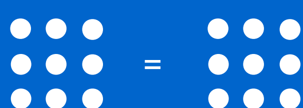
A MARS?

- What kind of representational keys?
  - Terminological?
- What kind of configurations?
  - Comprehensive?

Interpreting with MARS in mind…

An instance of the theorem:

$$1 + 3 + 5 = 9$$



The proof ...

- One thing is to express a theorem, and another to see that it is true!
- How can the theorem be proved?
  - Mathematical Induction?
  - Does mathematical induction depend on the notation?

Semi-formal inductive proof

Theorem:

$$1 + 3 + 5 + ... + (2n\text{-}1) = n^2$$

Proof by mathematical induction:

(1) $1 + 3 + 5 + ... + (2n\text{-}1) = n^2$

(2) $1 + 3 + 5 + ... + (2n\text{-}1) + (2(n+1)\text{-}1) = n^2 + (2(n+1)\text{-}1)$

(3) $1 + 3 + 5 + ... + (2n\text{-}1) + (2(n+1)\text{-}1) = n^2 + 2n+1$

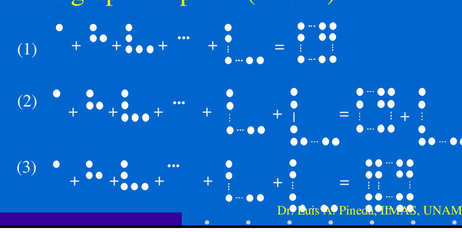
(4) $1 + 3 + 5 + ... + (2n\text{-}1) + (2(n+1)\text{-}1) = (n+1)^2$

From (2) to (4) intermediate manipulations!

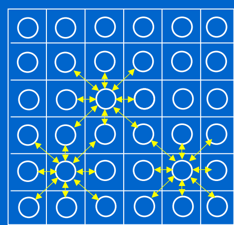
Inductive graphical reasoning

Theorem:

Inductive graphical proof (direct!):



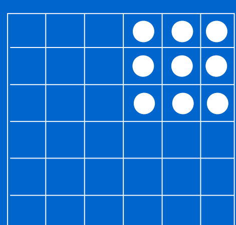
A rectangular retina



Inductive proof by visualization

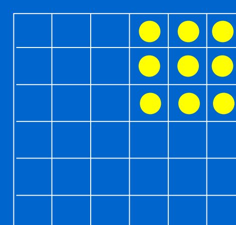
- (1) Visualize theorem's instance
  - Visualize object with square notation
  - Visualize object as a sequence of L´s
  - Verify identity of both visualizations
- (2) Apply inductive hypothesis
- (3) Repeat (1)

Inductive proof by visualization



Express the theorem

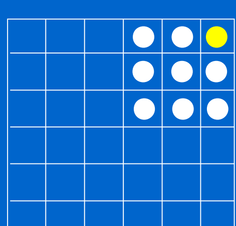
Inductive proof by visualization



Express the theorem

Visualize a square

Inductive proof by visualization

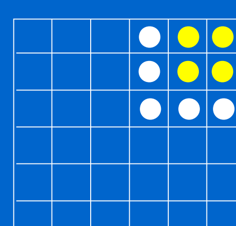


Express the theorem

Visualize a square

Visualize a sequence of L´s

Inductive proof by visualization

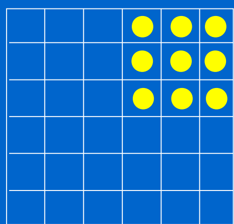


Express the theorem

Visualize a square

Visualize a sequence of L´s

Inductive proof by visualization



Express the theorem

Visualize a square

Visualize a sequence of L´s

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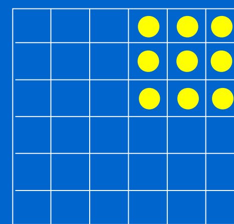
Inductive proof by visualization



Express the theorem

Visualize a square

They are the same!

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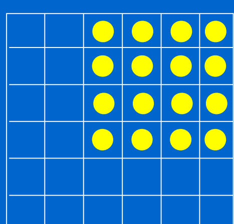
Inductive proof by visualization



Express the theorem

Visualize a square

Visualize a sequence of L´s

Visualize a square of $n + 1$

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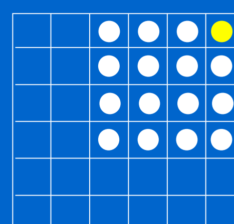
Inductive proof by visualization



Express the theorem

Visualize a square

Visualize a sequence of L´s

Visualize a square of $n + 1$

Visualize a sequence of L´s of $n + 1$

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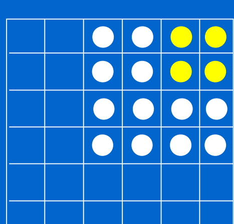
Inductive proof by visualization



Express the theorem

Visualize a square

Visualize a sequence of L´s

Visualize a square of $n + 1$

Visualize a sequence of L´s of $n + 1$

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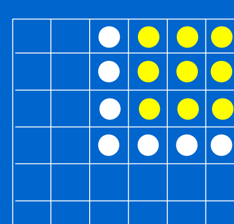
Inductive proof by visualization



Express the theorem

Visualize a square

Visualize a sequence of L´s

Visualize a square of $n + 1$

Visualize a sequence of L´s of $n + 1$

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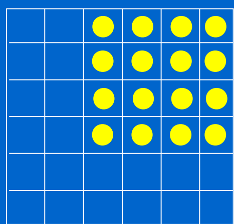
Inductive proof by visualization



Express the theorem

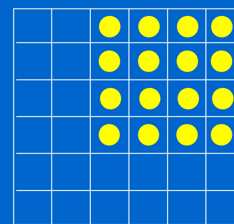
Visualize a square

Visualize a sequence of L´s

Visualize a square of $n + 1$

Visualize a sequence of L´s of $n + 1$

Inductive proof by visualization



Express the theorem

Visualize a square

Visualize a sequence of L´s

Visualize a square of $n + 1$

Visualize a sequence of L´s of $n + 1$

They are the same!

Only…a MARS?

- Is the representation only a MARS?
- Is there any abstraction in the notation?
- Do we really need the inductive argument?

Is there any abstraction in the notation?

Syntax

Semantics

1

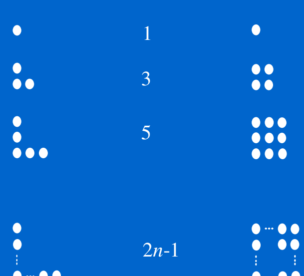
3

5

$2n - 1$

Abstraction in L´s and square's monadic notations

Syntax **Semantics** **Syntax** **Semantics**



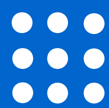
Semantics	Semantics
1	1
3	4
5	9
$2n$-1	n^2

A DUARS?

- What kind of representational keys?
  - Notational:
    - Any L-shape of size n stands for the number $2n - 1$
    - Any square of size n stands for the number n^2
- What kind of configurations?
  - Non-comprehensive: a configuration stands for an infinite number of cases
- One visualization is enough

Interpreting with DUARS in mind!

The theorem: the sum



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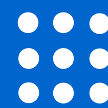
Interpretation change

The theorem: n^2



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Content

- Diagrammatic Proofs
- Representation of diagrams
- Abstraction and diagrams
- Pragmatics and theorem proving
- Diagrammatic theorem-proving
- Learning and graphical proof

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Learning and Visualization

- Pragmatics and theorem-proving
  - Reasoning *about* the system and not only *within* the system
  - Selecting notation and representational media
  - Interpretation change

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Learning and Visualization

- Proof of the theorem
  - Fix the appropriate representation and interpretation conventions, and *read off* de truth of the theorem!
- Discovering the theorem
  - Visualization in the context of learning

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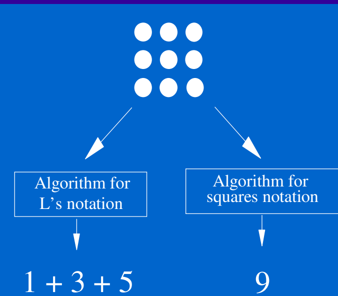
Computational re-interpretation



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Learning induction

- An inference *within* the system (traditional)
  - Generalization out of concrete instances?
  - Analogies expressed through the same kind of representational system?

Reasoning about medium and notation

- Meaning relations:
  - If the meaning of a expression interpreted according to one notational system is the same as the meaning of the same expression interpreted on the light of a different notation, there is an interesting relation between the two notational systems involved

Consequences of notation and medium

- Universal-meaning relation:
  - If the expressions are abstractions over an infinite number of cases, there is a very interesting relation between the two notational systems involved!

Diagrammatic induction

- An inference *about* the system
  - Reasoning about the notation
  - Reasoning about the representational media
  - To induce a *universal-meaning relation between notational keys* out of the regularities of medium and notation
  - A kind of synthetic truth?

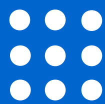
Reasoning about medium and notation

- Principle of interpretation of *composite area*:
  - The interpretation of a composite area is a function of the interpretation of its simple (no overlapping) parts

Diagrammatic Learning

- Given Notational Keys:
  - Any L-shape of size n stands for the number $2n - 1$
  - Any square of size n stands for the number n^2
- General property of the medium:
  - Any sequence or L-shapes aligned in relation to a reference point (the right-top corner) makes a square
- Principle of interpretation of composite area:
  - Any sequence of aligned L-shapes stands for the sum of the numbers represented by each of the L-shapes in the sequence (the function is the sum).
- Induced universal-meaning relation:
  - The sum of the first n odd numbers is n^2

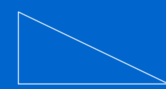
Alternative interpretations



$1 + 2 + 3 + \ldots + (n-1) + n + (n-1) + \ldots + 2 + 1 = n^2$

- Different notation
- Different relation between notation and medium
- Different theorem

Theorem of Pythagoras
(Bronowski en *The Ascent of Man*)



A language of square triangles

Notational Key

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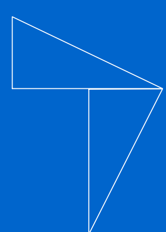
Theorem of Pythagoras



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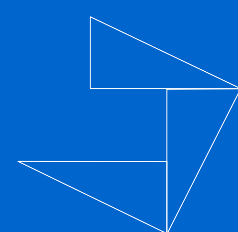
Theorem of Pythagoras



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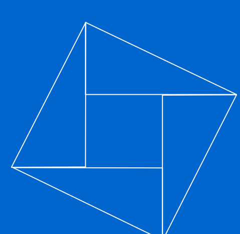
Theorem of Pythagoras



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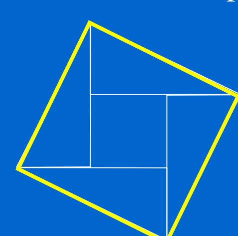
Theorem of Pythagoras

Visualization and reinterpretation!



Universal Meaning Relation

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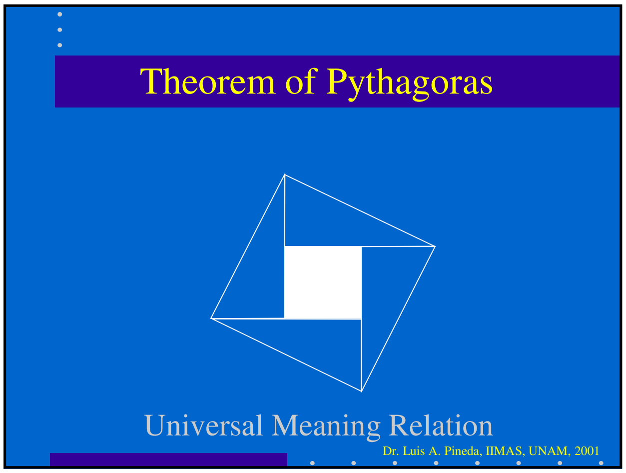
Theorem of Pythagoras
Universal Meaning Relation
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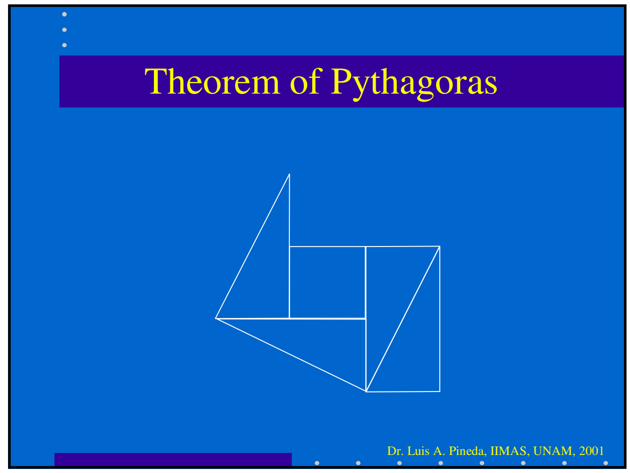
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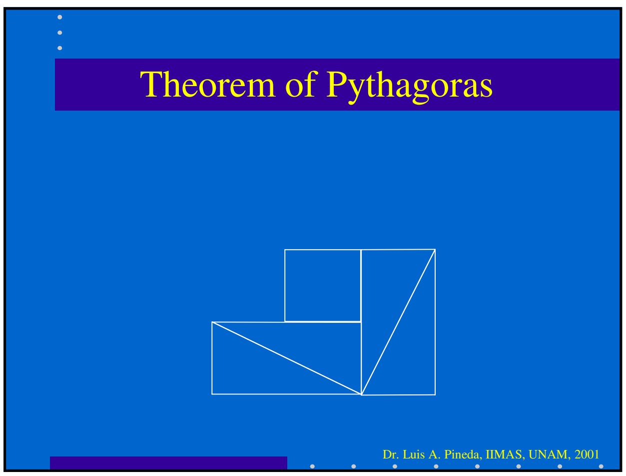
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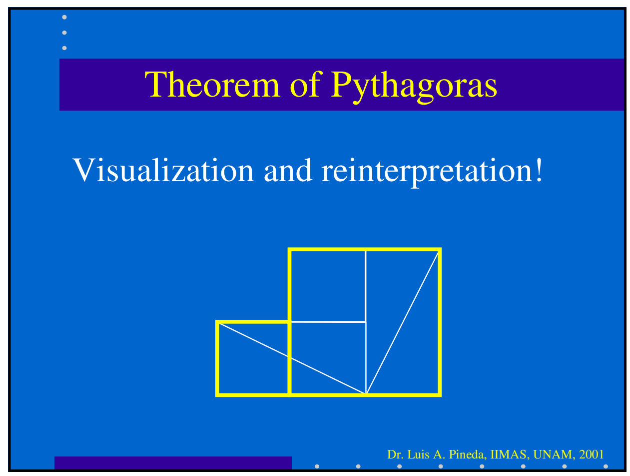
Theorem of Pythagoras
Visualization and reinterpretation!
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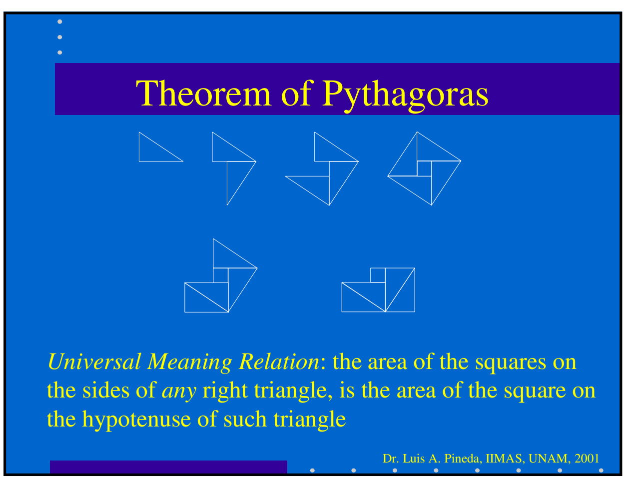
Theorem of Pythagoras
Universal Meaning Relation: the area of the squares on the sides of *any* right triangle, is the area of the square on the hypotenuse of such triangle
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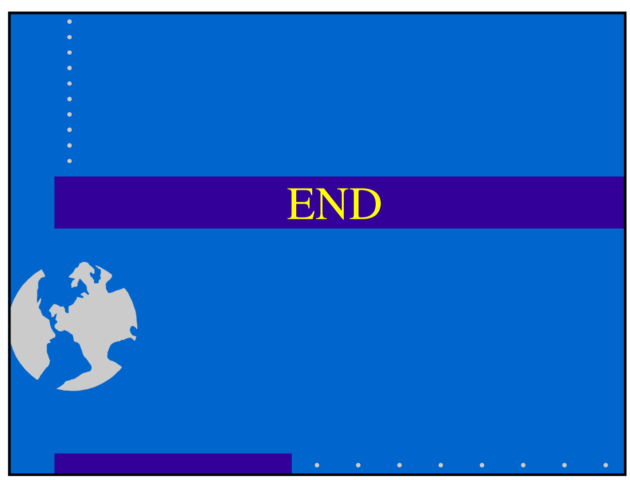
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