Equivalencia entre autómatas finitos deterministas e indeterministas

Alfabeto Σ: conjunto (de *símbolos*) finito, no vacío.

Cadena (de elementos de Σ): sucesión finita de elementos de Σ. Se representa yuxtaponiendo sus elementos. A la cadena de 0 símbolos se le llama **cadena vacía** y la representaremos con el símbolo ϵ ($\epsilon \notin \Sigma$, por supuesto).

$\Sigma^\star$: conjunto de (todas las) cadenas de elementos de Σ.

Lenguaje (sobre Σ): subconjunto de $\Sigma^\star$.

Autómata finito determinista: quintupleta $(Q, \Sigma, \delta, q_0, F)$, en que:

- Q es un conjunto finito de *estados*.
- Σ es un alfabeto ($Q \bigcap \Sigma = \emptyset$).
- $\delta : Q \times \Sigma \to Q$ es la *función de transición*.
- $q_0 \in Q$ es el estado *inicial*.
- $F \subseteq Q$ es el conjunto de *estados de aceptación*.

$$\begin{aligned}
&\hat{\delta} : Q \times \Sigma^\star \to Q, \\
&\hat{\delta}(q, \epsilon) = q, \\
&\hat{\delta}(q, wa) = \delta(\hat{\delta}(q, w), a).
\end{aligned}$$

$$\begin{aligned}
&M \text{ acepta a } w \iff \hat{\delta}(q_0, w) \in F. \\
&L(M) = \{w \in \Sigma^\star : M \text{ acepta a } w\}.
\end{aligned}$$

A.F.D.

$\delta_D : Q \times \Sigma \to Q,$

$\hat{\delta_D} : Q \times \Sigma^\star \to Q,$

$\hat{\delta_D}(q, \epsilon) = q, \qquad (\hat{\delta_D}.1)$

$\hat{\delta_D}(q, wa) = \delta_D(\hat{\delta_D}(q, w), a) \qquad (\hat{\delta_D}.2)$

M_D acepta $w \Leftrightarrow \hat{\delta_D}(q_0, w) \in F.$

$L(M_D) = \{w \in \Sigma^\star : M_D \text{ acepta } w\}.$

A.F.I.

$\delta_I : Q \times \Sigma \to 2^Q,$

$\hat{\delta}_I : Q \times \Sigma^\star \to 2^Q,$

$\hat{\delta}_I(q, \epsilon) = \{q\}, \qquad (\hat{\delta}_I.1)$

$$\hat{\delta}_I(q, wa) = \bigcup_{r \in \hat{\delta}_I(q,w)} \delta_I(r, a) \qquad (\hat{\delta}_I.2)$$

$$= \{p : \exists_r (r \in \hat{\delta}_I(q, w) \wedge p \in \delta_I(r, a))\}.$$

$\hat{\hat{\delta}}_I : 2^Q \times \Sigma^\star \to 2^Q,$

$$\hat{\hat{\delta}}_I(P, w) = \bigcup_{q \in P} \hat{\delta}_I(q, w) \qquad (\hat{\hat{\delta}}_I)$$

$$= \{p : \exists_q (q \in P \wedge p \in \hat{\delta}_I(q, w))\}.$$

M_I acepta $w \Leftrightarrow \hat{\delta}_I(q_0, w) \cap F \neq \emptyset.$

$L(M_I) = \{w \in \Sigma^\star : M_I \text{ acepta } w\}.$

Propiedades

$\hat{\delta}_I(p, a) = \delta_I(p, a). \qquad (1)$

$\hat{\hat{\delta}}_I(\{p\}, w) = \hat{\delta}_I(p, w). \qquad (2)$

$\hat{\hat{\delta}}_I(P, wa) = \hat{\hat{\delta}}_I(\hat{\hat{\delta}}_I(P, w), a). \qquad (3)$

$\hat{\delta}_I(q, wa) = \hat{\hat{\delta}}_I(\hat{\delta}_I(q, w), a). \qquad (4)$

(1) y (2) se demuestran fácilmente.

Prueba de (3):

$$
\begin{aligned}
\hat{\hat{\delta}}_I(P, wa) &\overset{(\hat{\hat{\delta}}_I)}{=} \{p : \exists_q (q \in P \wedge p \in \hat{\delta}_I(q, wa))\} \\
&\overset{(\hat{\delta}_I.2)}{=} \{p : \exists_q (q \in P \wedge \exists_r (r \in \hat{\delta}_I(q, w) \wedge p \in \delta_I(r, a)))\} \\
&= \{p : \exists_{q,r} (q \in P \wedge r \in \hat{\delta}_I(q, w) \wedge p \in \delta_I(r, a))\}.
\end{aligned}
$$

$$
\begin{aligned}
\hat{\hat{\delta}}_I(\hat{\hat{\delta}}_I(P, w), a) &\overset{(\hat{\hat{\delta}}_I)}{=} \{p : \exists_r (r \in \hat{\hat{\delta}}_I(P, w) \wedge p \in \hat{\delta}_I(r, a))\} \\
&\overset{(\hat{\hat{\delta}}_I, 1)}{=} \{p : \exists_r (\exists_q (q \in P \wedge r \in \hat{\delta}_I(q, w)) \wedge p \in \delta_I(r, a))\} \\
&= \{p : \exists_{r,q} (q \in P \wedge r \in \hat{\delta}_I(q, w) \wedge p \in \delta_I(r, a))\}. \ \square
\end{aligned}
$$

Prueba de (4):

$$
\begin{aligned}
\hat{\delta}_I(q, wa) &\overset{(2)}{=} \hat{\hat{\delta}}_I(\{q\}, wa) \\
&\overset{(3)}{=} \hat{\hat{\delta}}_I(\hat{\hat{\delta}}_I(\{q\}, w), a) \\
&\overset{(2)}{=} \hat{\hat{\delta}}_I(\hat{\delta}_I(q, w), a). \ \square
\end{aligned}
$$

<u>Teorema.</u> Para cada a.f.i.

$$M_I = (Q, \Sigma, \delta_I, q_0, F)$$

hay a.f.d. que acepta $L(M_I)$.

Demostración. Sea el a.f.d.

$$M'_D = (2^Q, \Sigma, \delta'_D, \{q_0\}, F'),$$

en que

$$\begin{array}{ll} \delta'_D(P,a) = \hat{\hat{\delta}}_I(P,a), & (\delta'_D) \\ f \in F' \Leftrightarrow f \cap F \neq \emptyset. & (F') \end{array}$$

<u>Lema.</u> $\hat{\delta'}_D(\{q_0\}, w) = \hat{\hat{\delta}}_I(\{q_0\}, w)$ $\left(\stackrel{(2)}{=} \hat{\delta}_I(q_0, w) \right)$.

Prueba (por inducción sobre $|w|$).

Caso 0)

$$\begin{array}{rcl} \hat{\delta'}_D(\{q_0\}, \epsilon) & \stackrel{(\hat{\delta_D}.1)}{=} & \{q_0\} \\ & \stackrel{(\hat{\delta_I}.1)}{=} & \hat{\delta}_I(q_0, \epsilon) \\ & \stackrel{(2)}{=} & \hat{\hat{\delta}}_I(\{q_0\}, \epsilon). \end{array}$$

Caso $n+1$)

$$\begin{array}{rcl} \hat{\delta'}_D(\{q_0\}, wa) & \stackrel{(\hat{\delta_D}.2)}{=} & \delta'_D(\hat{\delta'}_D(\{q_0\}, w), a) \\ & \stackrel{(h.i.)}{=} & \delta'_D(\hat{\hat{\delta}}_I(\{q_0\}, w), a) \\ & \stackrel{(\delta'_D)}{=} & \hat{\hat{\delta}}_I(\hat{\hat{\delta}}_I(\{q_0\}, w), a) \\ & \stackrel{(3)}{=} & \hat{\hat{\delta}}_I(\{q_0\}, wa). \ \Box \end{array}$$

Entonces $L(M'_D) = L(M_I)$, porque

$$\begin{array}{rcl} w \in L(M'_D) & \stackrel{(L(M_D))}{\Longleftrightarrow} & \hat{\delta'}_D(\{q_0\}, w) \in F' \\ & \stackrel{(F')}{\Longleftrightarrow} & \hat{\delta'}_D(\{q_0\}, w) \cap F \neq \emptyset \\ & \stackrel{(\text{lema})}{\Longleftrightarrow} & \hat{\delta}_I(q_0, w) \cap F \neq \emptyset \\ & \stackrel{(L(M_I))}{\Longleftrightarrow} & w \in L(M_I). \ \Box \end{array}$$

A.F.I. con transiciones ϵ

$$\delta_\epsilon : Q \times (\Sigma \cup \{\epsilon\}) \to 2^Q,$$

$$\epsilon\text{-cl} : Q \to 2^Q,$$

$$q \in \epsilon\text{-cl}(q), \qquad (q \overset{\epsilon}{\rightsquigarrow} q)$$

$$r \in \epsilon\text{-cl}(q) \wedge p \in \delta_\epsilon(r, \epsilon) \Rightarrow p \in \epsilon\text{-cl}(q). \qquad (q \overset{\epsilon}{\rightsquigarrow} p)$$

O sea que $\epsilon\text{-cl}(q) = \{p : q \overset{\epsilon}{\rightsquigarrow} p\}$ $\qquad (\epsilon\text{-cl})$

$$\epsilon\text{-CL} : 2^Q \to 2^Q,$$

$$\epsilon\text{-CL}(P) = \bigcup_{q \in P} \epsilon\text{-cl}(q) \qquad (\epsilon\text{-CL})$$

$$\hat{\delta}_\epsilon : Q \times \Sigma^\star \to 2^Q,$$

$$\hat{\delta}_\epsilon(q, \epsilon) = \epsilon\text{-cl}(q), \qquad (\hat{\delta}_\epsilon.1)$$

$$\hat{\delta}_\epsilon(q, wa) = \epsilon\text{-CL} \bigcup_{r \in \hat{\delta}_\epsilon(q,w)} \delta_\epsilon(r, a) \qquad (\hat{\delta}_\epsilon.2)$$

$$\hat{\hat{\delta}}_\epsilon : 2^Q \times \Sigma^\star \to 2^Q,$$

$$\hat{\hat{\delta}}_\epsilon(P, w) = \bigcup_{q \in P} \hat{\delta}_\epsilon(q, w) \qquad (\hat{\hat{\delta}}_\epsilon)$$

M_ϵ acepta $w \Leftrightarrow \hat{\delta}_\epsilon(q_0, w) \cap F \neq \emptyset$.

$L(M_\epsilon) = \{w \in \Sigma^\star : M_\epsilon \text{ acepta } w\}$.

Propiedades $\quad (q \overset{w}{\rightsquigarrow} r$ abrevia $r \in \hat{\delta}_\epsilon(q, w))$

$$q \overset{w}{\rightsquigarrow} r \Rightarrow \epsilon\text{-cl}(r) \subseteq \hat{\delta}_\epsilon(q, w) \qquad (q \overset{w}{\rightsquigarrow} r \wedge r \overset{\epsilon}{\rightsquigarrow} s \Rightarrow q \overset{w}{\rightsquigarrow} s) \qquad (5)$$

$$\epsilon\text{-CL} \bigcup_{k \in K} P_k = \bigcup_{k \in K} \epsilon\text{-CL}(P_k). \qquad (6)$$

$$\hat{\delta}_\epsilon(q, wa) = \hat{\hat{\delta}}_\epsilon(\hat{\delta}_\epsilon(q, w), a). \qquad (7)$$

$$\bigcup_{r \in \hat{\delta}_\epsilon(q,w)} \bigcup_{s \in \epsilon\text{-cl}(r)} \delta_\epsilon(s, a) = \bigcup_{r \in \hat{\delta}_\epsilon(q,w)} \delta_\epsilon(r, a) \qquad (8)$$

Prueba de (6):

$$p \in \epsilon\text{-CL} \bigcup_{k \in K} P_k$$
$$\stackrel{(\epsilon\text{-CL})}{\Longrightarrow} \exists_q (q \in \bigcup_{k \in K} P_k \wedge p \in \epsilon\text{-cl}(q))$$
$$\Longrightarrow \exists_q (\exists_{k \in K} . q \in P_k \wedge p \in \epsilon\text{-cl}(q))$$
$$\Longrightarrow \exists_{k \in K} \exists_q (q \in P_k \wedge p \in \epsilon\text{-cl}(q))$$
$$\stackrel{(\epsilon\text{-CL})}{\Longrightarrow} \exists_{k \in K} (p \in \epsilon\text{-CL}(P_k))$$
$$\Longrightarrow p \in \bigcup_{k \in K} \epsilon\text{-CL}(P_k).$$

$$p \in \bigcup_{k \in K} \epsilon\text{-CL}(P_k)$$
$$\Longrightarrow \exists_{k \in K} (p \in \epsilon\text{-CL}(P_k))$$
$$\Longrightarrow p \in \epsilon\text{-CL} \bigcup_{k \in K} P_k. \ \Box$$

Prueba de (7):

$$\hat{\hat{\delta}}_\epsilon(\hat{\delta}_\epsilon(q, w), a)$$
$$\stackrel{(\hat{\hat{\delta}}_\epsilon)}{=} \bigcup_{r \in \hat{\delta}_\epsilon(q,w)} \hat{\delta}_\epsilon(r, a)$$
$$\stackrel{(\hat{\delta}_\epsilon)}{=} \bigcup_{r \in \hat{\delta}_\epsilon(q,w)} \epsilon\text{-CL} \bigcup_{s \in \epsilon\text{-cl}(r)} \delta_\epsilon(s, a)$$
$$\stackrel{(6)}{=} \epsilon\text{-CL} \bigcup_{r \in \hat{\delta}_\epsilon(q,w)} \bigcup_{s \in \epsilon\text{-cl}(r)} \delta_\epsilon(s, a)$$
$$\stackrel{(8)}{=} \epsilon\text{-CL} \bigcup_{r \in \hat{\delta}_\epsilon(q,w)} \delta_\epsilon(r, a)$$
$$\stackrel{(\hat{\delta}_\epsilon.2)}{=} \hat{\delta}_\epsilon(q, wa). \ \Box$$

Prueba de (8):

$\subseteq$)

1. $p \in \bigcup_{r \in \hat{\delta}_\epsilon(q,w)} \bigcup_{s \in \epsilon\text{-cl}(r)}$ $[Hip]$
2. $\exists_{r,s} (r \in \hat{\delta}_\epsilon(q, w) \wedge s \in \epsilon\text{-cl}(r) \wedge p \in \delta_\epsilon(s, a))$ $[1]$
3. $q \overset{w}{\rightsquigarrow} r \wedge r \overset{\epsilon}{\rightsquigarrow} s$ $[2]$
4. $q \overset{w}{\rightsquigarrow} s$ $[3, (5)]$
5. $s \in \hat{\delta}_\epsilon(q, w)$ $[4]$
6. $p \in \bigcup_{s \in \hat{\delta}_\epsilon(q,w)} \delta_\epsilon(s, a)$ $[2, 5]$

$\supseteq$)

1. $p \in \bigcup_{r \in \hat{\delta}_\epsilon(q,w)} \delta_\epsilon(r, a)$ $[Hip]$
2. $\exists_r (r \in \hat{\delta}_\epsilon(q, w) \wedge p \in \delta_\epsilon(r, a))$ $[1]$
3. $r \in \epsilon\text{-cl}(r)$ $[\epsilon\text{-cl}]$
4. $p \in \bigcup_{s \in \epsilon\text{-cl}(r)} \delta_\epsilon(s, a)$ $[1, 3]$
5. $p \in \bigcup_{r \in \hat{\delta}_\epsilon(q,w)} \bigcup_{s \in \epsilon\text{-cl}(r)} \delta_\epsilon(s, a)$ $[4]$ $\Box$

<u>Teorema.</u> Para cada a.f.i.-ϵ

$$M_\epsilon = (Q, \Sigma, \delta_\epsilon, q_0, F)$$

hay a.f.i. que acepta $L(M_\epsilon)$.

Demostración. Sea el a.f.i.

$$M_I = (Q, \Sigma, \delta_I, q_0, F'),$$

en que

$$\delta_I(q,a) = \hat{\hat{\delta}}_\epsilon(q,a), \qquad (\delta_I)$$

$$F' = \begin{cases} F \cup \{q_0\}, \text{ si } \epsilon\text{-cl}(q_0) \cap F \neq \emptyset; \\ F, \qquad \text{en caso contrario.} \end{cases}$$

<u>Lema.</u> $\hat{\delta}_I(q_0,w) = \hat{\hat{\delta}}_\epsilon(q_0,w)$, $(w \neq \epsilon)$.

Prueba (por inducción sobre $|w|$).

Caso $|w| = 1$)

$$\begin{aligned} \hat{\delta}_I(q_0,a) &\overset{(1)}{=} \delta_I(q_0,a) \\ &\overset{(\delta_I)}{=} \hat{\hat{\delta}}_\epsilon(q_0,a). \end{aligned}$$

Caso $|w| > 1$)

$$\begin{aligned} \hat{\delta}_I(q_0,wa) &\overset{(4)}{=} \hat{\hat{\delta}}_I(\hat{\delta}_I(q_0,w),a) \\ &\overset{(h.i.)}{=} \hat{\hat{\delta}}_I(\hat{\hat{\delta}}_\epsilon(q_0,w),a) \\ &\overset{(\hat{\hat{\delta}}_I)}{=} \bigcup_{q \in \hat{\hat{\delta}}_\epsilon(q_0,w)} \hat{\delta}_I(q,a) \\ &\overset{(1)}{=} \bigcup_{q \in \hat{\hat{\delta}}_\epsilon(q_0,w)} \delta_I(q,a) \\ &\overset{(\delta_I)}{=} \bigcup_{q \in \hat{\hat{\delta}}_\epsilon(q_0,w)} \hat{\hat{\delta}}_\epsilon(q,a) \\ &\overset{(\hat{\hat{\hat{\delta}}}_\epsilon)}{=} \hat{\hat{\hat{\delta}}}_\epsilon(\hat{\hat{\delta}}_\epsilon(q_0,w),a) \\ &\overset{(7)}{=} \hat{\hat{\delta}}_\epsilon(q_0,wa). \ \square \end{aligned}$$

Entonces $L(M_I) = L(M_\epsilon)$, porque

$$\begin{aligned} w \in L(M_I) &\overset{(L(M_I))}{\Longleftrightarrow} \hat{\delta}_I(q_0,w) \cap F' \neq \emptyset \\ &\ddots \\ &\Longleftrightarrow \hat{\hat{\delta}}_\epsilon(q_0,w) \cap F \neq \emptyset \\ &\overset{(L(M_\epsilon))}{\Longleftrightarrow} w \in L(M_\epsilon). \ \square \end{aligned}$$

(Demostración completa en página siguiente.)

Prueba de $w \in L(M_I) \Leftrightarrow w \in L(M_\epsilon)$, es decir,
$\hat{\delta}_I(q_0, w) \cap F' \neq \emptyset \Leftrightarrow \hat{\delta}_\epsilon(q_0, w) \cap F \neq \emptyset$.

Caso $|w| = 0$.
$\Rightarrow$) (por contraposición.)

$$\begin{aligned}
&\hat{\delta}_\epsilon(q_0, \epsilon) \cap F = \emptyset \\
&\overset{(\hat{\delta}_\epsilon.1)}{\Longrightarrow} \quad \epsilon\text{-cl}(q_0) \cap F = \emptyset \\
&\overset{(F',\, q_0 \in \epsilon\text{-cl}(q_0))}{\Longrightarrow} \quad F' = F \wedge q_0 \notin F \\
&\Longrightarrow \quad q_0 \notin F' \\
&\overset{(\hat{\delta}_I.1)}{\Longrightarrow} \quad \hat{\delta}_I(q_0, \epsilon) \cap F' = \emptyset.
\end{aligned}$$

$\Leftarrow$)

$$\begin{aligned}
&\hat{\delta}_\epsilon(q_0, \epsilon) \cap F \neq \emptyset \\
&\overset{(F',\, \hat{\delta}_\epsilon.1)}{\Longrightarrow} \quad F' = F \cup \{q_0\} \\
&\Longrightarrow \quad q_0 \in F' \\
&\overset{(\hat{\delta}_I.1)}{\Longrightarrow} \quad \hat{\delta}_I(q_0, \epsilon) \cap F' \neq \emptyset.
\end{aligned}$$

En la siguiente deducción, las hipótesis 2 y 6 son exhaustivas, corresponden a las dos posibilidades para definir F'. De igual manera, dentro del **caso B** las hipótesis 10 y 12 son exhaustivas, surgen al considerar las alternativas que se presentan en 9.

Caso $|w| > 0$.
$\Rightarrow$)

1. $\hat{\delta}_I(q_0, w) \cap F' \neq \emptyset$ [Hip]

Caso A

2. $\epsilon\text{-cl}(q_0) \cap F = \emptyset$ [hip]
3. $F' = F$ [$2, F'$]
4. $\hat{\delta}_I(q_0, w) \cap F \neq \emptyset$ [$1, 3$]
5. $\hat{\delta}_\epsilon(q_0, w) \cap F \neq \emptyset$. [$4$, lema]

Caso B

6. $\epsilon\text{-cl}(q_0) \cap F \neq \emptyset$ [hip]
7. $F' = F \cup \{q_0\}$ [$6, F'$]
8. $(\hat{\delta}_I(q_0, w) \cap F) \cup (\hat{\delta}_I(q_0, w) \cap \{q_0\}) \neq \emptyset$ [$2, 7$]
9. $\hat{\delta}_I(q_0, w) \cap F \neq \emptyset \vee \hat{\delta}_I(q_0, w) \cap \{q_0\} \neq \emptyset$ [8]

subcaso B-1

10. $\hat{\delta}_I(q_0, w) \cap F \neq \emptyset$ [hip]
11. $\hat{\delta}_\epsilon(q_0, w) \cap F \neq \emptyset$. [$10$, lema]

subcaso B-2

12. $\hat{\delta}_I(q_0, w) \cap \{q_0\} \neq \emptyset$ [hip]
13. $\hat{\delta}_\epsilon(q_0, w) \cap \{q_0\} \neq \emptyset$ [12, lema]
14. $q_0 \in \hat{\delta}_\epsilon(q_0, w)$ $(q_0 \overset{w}{\rightsquigarrow} q_0)$ [13]
15. $\exists_{f \in F}(q_0 \overset{\epsilon}{\rightsquigarrow} f)$ [6]
16. $\exists_{f \in F}(f \in \hat{\delta}_\epsilon(q_0, w))$ [$14, 15, (5)$]
17. $\hat{\delta}_\epsilon(q_0, w) \cap F \neq \emptyset$. [16]

$\Leftarrow$)

1. $\hat{\delta}_\epsilon(q_0, w) \cap F \neq \emptyset$ [Hip]
2. $\hat{\delta}_I(q_0, w) \cap F \neq \emptyset$ [1, lema]
3. $F \subseteq F'$ [F']
4. $\hat{\delta}_I(q_0, w) \cap F' \neq \emptyset$. [$2, 3$] $\square$