Beacon Coverage in Orthogonal Polyhedra

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Abstract

We consider a variant of the Art Gallery Problem in orthogonal polygons and orthogonal polyhedra using beacons as guards. A beacon is a device that attracts objects toward itself within a given domain. A beacon b covers a point if when a beacon attracts it, it reaches b. In this paper, we prove that there exist orthogonal polyhedra whose exterior cannot be covered even if we place a beacon at each of its vertices.

We also study the beacon coverage problem in orthogonal polyhedra, by extending the notion of vertex beacons to edge beacons. We prove that $\lfloor \frac{e}{12} \rfloor$ edge beacons are always sufficient while $\lfloor \frac{e}{21} \rfloor$ edge beacons are sometimes necessary to cover any orthogonal polyhedron. We also prove that $\lfloor \frac{e}{6} \rfloor$ edge beacons are always sufficient to cover simultaneously the interior and the exterior of any orthogonal polyhedron.

1 Introduction

A *beacon* is a fixed point in a polyhedron P that can induce a magnetic pull toward itself over all points in P. When a beacon b is activated, points in P move greedily to decrease their euclidean distance to b. A point p can move along any obstacles it hits on its way to a beacon b as long as its distance to b keeps on decreasing. Thus, the path from the initial position of p to a beacon b may alternate between moving in straight line segments contained in the interior of P and line segments on the faces of P.

The piecewise linear path created by the movement of p under the attraction of b is called the *attraction path* of p with respect to b. If the attraction path of p ends in b, we say that p is *covered* by b. If p is in a position where it is unable to move in such a way that its distance to b decreases, we say that it is 'stuck' and it has reached a local minimum, or dead end, see Figure 1.

Beacon attraction was introduced by Biro et al. [3, 4, 5]. This model extends the classical notion of visibility; if an object p is visible from a beacon q, then p moves towards q along the straight line segment joining p to q.

In this paper we consider two beacon *coverage* problems in orthogonal polygons and orthogonal polyhedra. The beacon coverage problem asks for a minimum set B of beacons placed in a domain P, in such a way that any point $p \in P$ is covered by at least one element of B. We then study the interior-exterior beacon coverage problem in which we ask for a minimum set B of beacons placed on the boundary of a domain P, in such a way that any point $p \in P$ and any point $p' \notin P$ are covered by at least one element of B.

2 Preliminaries

Let P be an orthogonal polygon on the plane. An edge e of P is a *right edge* if there is an $\varepsilon > 0$ such that any point at distance less than or equal to ε from any interior point of e and to the left of e belongs to the interior of P. *Left*, *top* and *bottom* edges are defined similarly.

A *polyhedron* in $\mathbb{R}^3$ is a compact connected set bounded by a piecewise linear 2-manifold. A *face* of a polyhedron is a maximal planar subset of its boundary whose interior is connected and non-empty. A polyhedron is *orthogonal* if all of its faces are parallel to the $\mathcal{XY}$, $\mathcal{XZ}$ or $\mathcal{YZ}$ planes. An *edge* is a minimal positive-length straight line segment shared by two faces and joining two vertices of the polyhedron. Each edge, with its two adjacent faces, determines a dihedral angle, internal to the polyhedron. In an orthogonal polyhedron each such angle is of either 90^o (at a *convex edge*) or 270^o (at a *reflex edge*).

An $\mathcal{X}$*-plane* is a plane that is perpendicular to the $\mathcal{X}$-axis; we define a $\mathcal{Y}$*-plane* and a $\mathcal{Z}$*-plane* in a similar way. An $\mathcal{X}$*-face* is a face of a polyhedron that is contained in an $\mathcal{X}$*-plane*; we define a $\mathcal{Y}$*-face* and a $\mathcal{Z}$*-face* in a similar way.

A $\mathcal{Y}$-face f of an orthogonal polyhedron P is a *left face* (*right face*), if for any interior point $q \in f$ there is an $\varepsilon > 0$ such that any point at distance less than or equal to ε from q and to the right (left) of f belongs to

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the interior of P. In a similar way, $\mathcal{Z}$-faces are classified into *top* or *bottom* faces, and $\mathcal{X}$-faces, as *front* or *back* faces.

A *connected polyhedron* P is a *lifting polyhedron* if there exists a $\mathcal{Z}$*-plane* Q such that for all planes parallel to Q their intersection with P is either empty, or it is a vertical translation of the intersection of P with Q.

The next definition was given by Damian et al. in [7]. An *orthotree* P is an orthogonal polyhedron made of cuboids glued face to face, such that the dual graph of P is a tree. The intersection of two adjacent cuboids in P is a 2-dimensional face of both cuboids, namely, a non-degenerate rectangle.

In this paper we consider two models of beacon attraction, vertex beacons and edge beacons. In the *vertex beacon model*, we place point beacons on the vertices of P. In the *edge beacon* model, we place a point beacon at each point of a closed edge e of P. We call e an *edge beacon*. When an edge beacon b is activated in an orthogonal polyhedron, an object p always moves towards the point $q \in b$ closest to p. If p reaches q we say that b *covers* p. Observe that if q is not an endpoint of b, the attraction path of p to q is contained in the plane β orthogonal to b that contains q. Therefore, in this case the attraction path of p with respect to b is as in $\mathbb{R}^2$, considering q as the beacon and $P \cap \beta$ as the polygon.

Consider the connected component S of $\overline{pq} \cap P$ that contains p. If S contains other points different from p, then p continues moving to q along $\overrightarrow{pq}$. Otherwise, p hits ∂P and there are three cases: (i) If p hits a vertex v, then p gets stuck at v. (ii) If p hits a point x in the interior of an edge e, and the orthogonal projection q_e of q over the straight line that contains e is different from p, then p moves along $\overrightarrow{pq_e}$. Otherwise, p gets stuck at x. (iii) If p hits a point x in the interior of a face f, and the projection q_f of q over the plane that contains f is different from p, then p moves along $\overrightarrow{pq_f}$. Otherwise, p gets stuck at x.

Figure 1 shows two examples of points reaching local minima on their way to vertex and edge beacons.

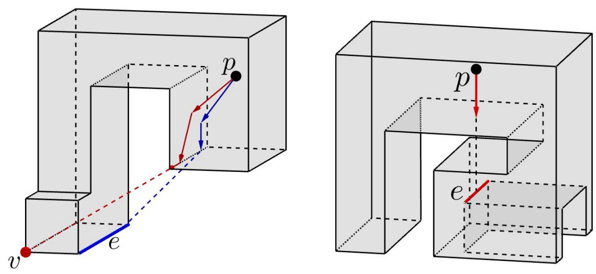


Figure 1: Two examples of points that reach a local minimum: (a) The attraction path of a point with respect to a vertex beacon and an edge beacon, both unreachable, and (b) the point gets stuck on its way to an edge beacon.

3 Covering orthogonal polygons

Bae et al. [2] proved that the interior of any orthogonal n-gon can be covered with $\lfloor \frac{n}{6} \rfloor$ vertex beacons. We consider now the problem of simultaneously covering the interior and exterior of orthogonal polygons.

Theorem 1 *Let P be an orthogonal polygon (possibly with holes) with n vertices. Then $\lfloor \frac{n}{4} \rfloor + 1$ vertex beacons are always sufficient to simultaneously cover the interior and the exterior of P.*

Proof. Suppose w.l.o.g. that there are at most $\lfloor \frac{n}{4} \rfloor$ right edges of P. Let B be the set of bottom vertices of the right edges of P. We place a beacon on each $b \in B$. We will prove that this set of beacons covers P. For each $b \in B$ consider the maximal vertical line segment s_b that contains b and is contained in P. Note that the set of line segments $S = \{s_b : b \in B\}$ divides P into histograms, such that in each histogram: There is only one right edge, and all top and bottom edges are contained in edges of P (since we only use vertical line segments to divide P), see Figure 2.

Let p be a point in a histogram H with $H \subset P$ and let b_h be one of the vertices of B that lie in the right edge of H. We will prove only the case when p is on or above b_h, the proof for the other case is symmetric. We claim that p is covered by the beacon placed in b_h. If p is on the right edge of H, we are done. Suppose p is above and to the left of b_h. Since H has only one right edge, the attraction path T of p with respect to b_h can only hit bottom edges of H. Since all the bottom edges of H are contained in edges of P, it follows that T is contained in H. If T does not finish at b_h, then it reaches a local minimum in a bottom edge of H. This local minimum has to be exactly above b_h, which is impossible because b_h is contained in the unique right edge of H. Therefore, these beacons cover the interior of P.

Now we prove that the beacons placed on the elements of B plus an extra beacon cover the exterior of P. Let R be a rectangle containing P in its interior. Let $P' = R \setminus int(P)$, where $int(P)$ denotes the interior of P. Note that the elements in B are bottom vertices of left edges of P'. As before, we can use the same technique to cover the interior of P' with beacons placed on the elements of B plus an extra beacon placed on the bottom vertex v_R of a leftmost edge of R. We will prove that this beacon can be replaced by a beacon placed on the bottom vertex v_l of the leftmost edge of P.

Let $H_R \subset P'$ be the histogram whose left edge is the left edge of R. Let p be a point contained in H_R. If p is to the left of the vertical line ℓ trough v_l then we are done. We will prove only the case when p is to the right of ℓ, and above the horizontal line h trough v_l. The proof for the case when p is below h is symmetric.

Since H_R has only one left edge, the attraction path T of p with respect to v_l can only hit bottom edges of H_R. Since all the bottom edges of H_R are contained in edges of P except for the one that is contained in the bottom edge of R, T is contained in H_R. If T does not finish at v_l, then it reaches a local minimum in the interior of a bottom edge of H_R. This local minimum has to be exactly above v_l, which is impossible because v_l is contained in a leftmost edge of P. Therefore H_R is covered by the beacon placed at v_l. Hence the beacons in B together with v_l cover both the interior and the exterior of P. □

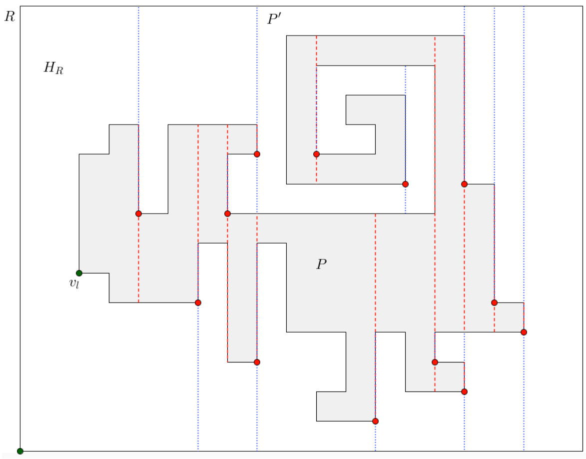


Figure 2: Regions obtained by the decomposition selecting the bottom vertices of the right edges of P.

4 Covering orthogonal polyhedra

It is known that not every polyhedron can be covered with vertex beacons [6], even if the polyhedron is orthogonal [1]. There exist well known families of orthogonal polyhedra whose interior can be covered with vertex beacons. Such is the case of the orthotrees [1]. However, there exists an orthotree polyhedron such that it is not possible to cover its exterior with vertex beacons. That is the case of the polyhedron shown in Figure 3a, that we describe next.

Our example is based on the octoplex polyhedron, proposed by T. S. Michael in [8]. The octoplex consists of a cuboid with six channels, each one of them going across a different face. It is known that the octoplex cannot be guarded with *vertex guards* (a vertex guard is a guard placed on a vertex). We take six notched beams arranged as the six channels of the octoplex. Then we join them by means of orthogonal pipes arranged properly to form an orthotree, see Figure 3a.

A point in the exterior of P that is not covered by any vertex of P is the "center" point p of the region enclosed by the beams of P. Consider the wedge (dihedral angle) whose axis is vertical and contains p. This wedge is delimited by the two edges at the end of the concavity of the notch of the red beam, see Figure 3b. Note that the beam divides the wedge into two connected regions: W_p which contains p and W'_p which does not. Following the notation of Figure 3c, the polyhedron is constructed with p at the origin so that q and v satisfy $\frac{q_y}{q_x} < \frac{v_y}{v_x}$, thus ensuring that the pq ray intersects the notch of the red beam. Therefore, any beacon placed in the interior of W'_p cannot cover p. Similarly, we construct a wedge for each beam in such a way that every vertex of P is in the interior of one of these wedges.

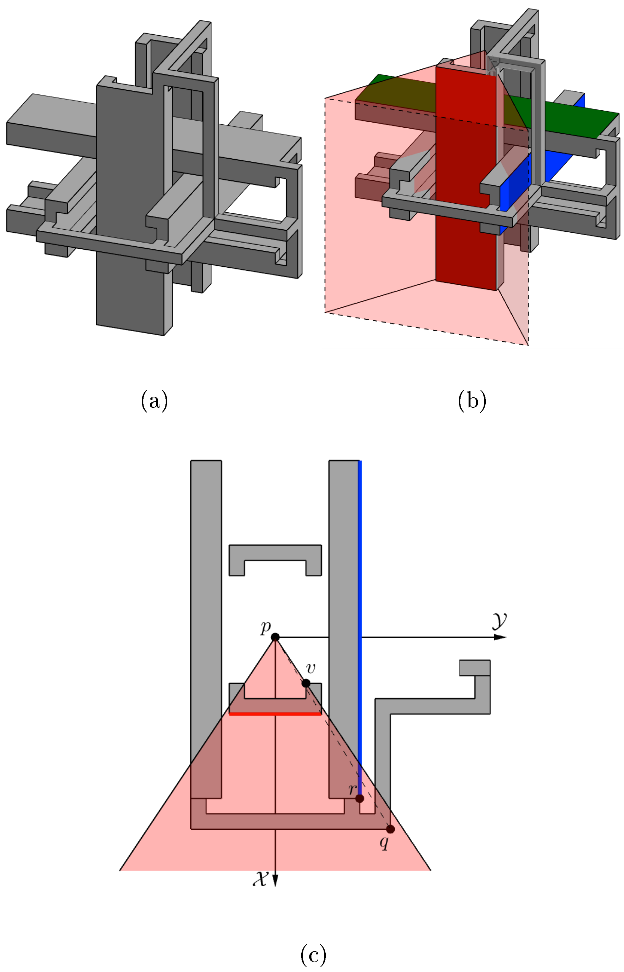


Figure 3: (a) An orthotree whose exterior cannot be covered with vertex beacons. (b) Wedge whose axis is vertical and contains the center point p. This wedge is delimited by the two edges at the end of the concavity of the notch of the red beam. (c) Orthogonal projection in the $\mathcal{XY}$ plane of some conveniently selected elements.

Since vertex beacons are not enough to cover orthogonal polyhedra, it is natural to study the edge beacon model. It is straightforward to see that if we place an edge beacon at each edge of a polyhedron P (orthogonal or not) these edge beacons always cover P.

Next, we prove that any orthogonal polyhedron with e edges can be covered with $\lfloor \frac{e}{12} \rfloor$ edge beacons and that sometimes $\lfloor \frac{e}{12} \rfloor$ edge beacons are necessary. We also prove that $\lfloor \frac{e}{6} \rfloor$ edge beacons are always sufficient to simultaneously cover the interior and exterior of any orthogonal polyhedron. In another paper we prove that any orthotree with n vertices can be guarded with at most $\lfloor \frac{n}{8} \rfloor$ vertex guards, and therefore covered using at most $\lfloor \frac{n}{8} \rfloor$ vertex beacons [1].

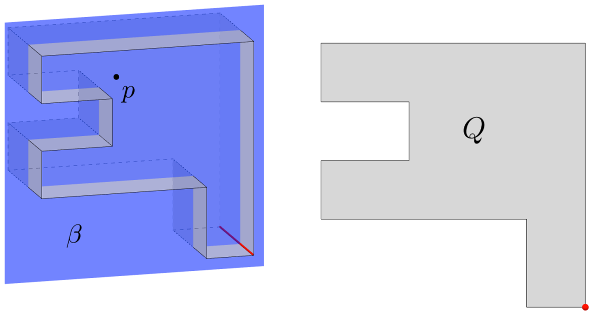
4.1 Covering orthogonal polyhedra with edge beacons

Now we define for each $F \in$ *{left, right, top, bottom, front, back}* and for each $E \in$ *{left, right, top, bottom, front, back}* the *F-E* rule. The *F-E* rule selects the E edges from the F faces of an orthogonal polyhedron P, seen from the outside. For example, the *right-bottom* rule selects the bottom edges of the right faces of P. Note that each face type contains only four different types of edges, namely, if $F =$ *front (or back)* then $E \in$ *{left, right, top, bottom}*, if $F =$ *top (or bottom)* then $E \in$ *{left, right, front, back}*, and if $F =$ *right (or left)* then $E \in$ *{front, back, top, bottom}*. Thus, a rule like the *top-bottom* rule selects no edges of P.

Lemma 2 *For every orthogonal polyhedron P there exists an* F-E *rule which selects at most $\lfloor \frac{e}{12} \rfloor$ edges from P, where e is the number of edges of P.*

Proof. Let A, B, C be the number of edges in the $\mathcal{Y}$, $\mathcal{X}$, and $\mathcal{Z}$ faces, respectively. Since $A+B+C = 2e$ one of A, B or C is at most $\lfloor \frac{2e}{3} \rfloor$. Then suppose w.l.o.g. that the set $F_\mathcal{Y}$ consisting of the $\mathcal{Y}$ faces has at most $\lfloor \frac{2e}{3} \rfloor$ edges of P. There are two kinds of faces in $F_\mathcal{Y}$: left and right. Let $R \subset F_\mathcal{Y}$ be the set of right faces of $F_\mathcal{Y}$ and let E_R be the set of edges that belong to faces of R. Suppose w.l.o.g. that $|E_R|$ is at most half of the number of edges belonging to the faces of $F_\mathcal{Y}$, i.e., $|E_R| \leq \lfloor \frac{e}{3} \rfloor$. There are four kinds of edges in E_R: top, bottom, front and back. Therefore one of these four types of edges has at most $\left\lfloor \frac{|E_R|}{4} \right\rfloor$ edges of P. Suppose w.l.o.g that the set of bottom edges of E_R has at most $\left\lfloor \frac{|E_R|}{4} \right\rfloor \leq \lfloor \frac{e}{12} \rfloor$ edges of P. Note that these are the edges selected by the *right-bottom* rule. In any other case the proof is analogous by selecting the appropriate *F-E* rule. □

Theorem 3 *Let P be an orthogonal polyhedron with e edges. Then $\lfloor \frac{e}{12} \rfloor$ edge beacons are always sufficient to cover P.*



(a) (b)

Figure 4: (a) The point p and the $\mathcal{X}-$plane β, (b) polygon Q.

Proof. By Lemma 2 we can suppose w.l.o.g. that the set B of edges selected by the *right-bottom* rule has at most $\lfloor \frac{e}{12} \rfloor$ edges.

We place an edge beacon on each $b \in B$. We will prove that this set of edge beacons covers P. Let p be a point in P. Let β be the $\mathcal{X}$-plane that contains p. Let Q be the connected component of $\beta \cap P$ that contains p, as shown in Figure 4. Note that Q is an orthogonal polygon, and that each bottom vertex of a right edge of Q is of the form $b \cap Q$ for some $b \in B$.

Since the attraction path of p with respect to an edge beacon remains in β, using the same reasoning as in the proof of Theorem 1, we can prove that p is covered by a beacon placed on an edge $b \in B$. □

Theorem 4 *There exists a family of orthogonal polyhedra with e edges, such that $\lfloor \frac{e}{21} \rfloor$ edge beacons are necessary to cover their interior.*

Proof. We construct a lifting polyhedron P, based on a rectangular spiral polygon consisting of a sequence of $r + 1$ thin rectangles, Figure 5 shows a top view of P.

Let $e_0, e_1, \ldots, e_r$ be a set of consecutive convex edges of P that are parallel to the $\mathcal{Z}$-axis, whose orthogonal projections are shown in Figure 5. Let $e'_1, e'_2, \ldots, e'_{r-1}$ be the set of consecutive reflex edges of P that are parallel to the $\mathcal{Z}$-axis, and e'_0 and e'_r be the convex edges of P parallel to the $\mathcal{Z}$-axis that have an incident face in common with e'_1 and e'_{r-1}, respectively. From the top, they correspond to the reflex and convex vertices of the projection of P on the $\mathcal{XY}$ plane, see Figure 5.

Suppose for the sake of simplicity that $r = 7m$ for $m \in \mathbb{N}$. For each $0 \leq k < m$, we place a distinguished point p_k in the interior of P near enough the center of the face formed by the edges e'_{7k} and e'_{7k+1}, and a distinguished point p'_k in the center of the rectangle formed by the edges e'_{7k+4} and e_{7k+4}, as shown in Figure 5. Note that p'_k is in a region that is not covered by e_{7k+2}, e'_{7k+2}, e_{7k+6} neither e'_{7k+6}.

Note that there is no edge covering two distinguished points at the same time. Since there are $\frac{2r}{7}$ distinguished points and P has $e = 6(r+1)$ edges, we need at least $\lfloor \frac{e}{21} \rfloor$ edge beacons to cover P. □

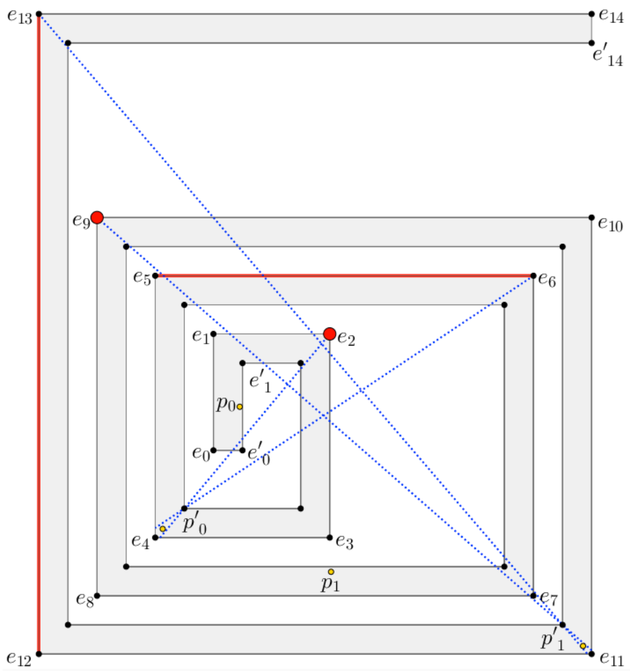


Figure 5: Orthogonal projection over the $\mathcal{XY}$ plane of a spiral polyhedron with e edges that requires $\lfloor \frac{e}{21} \rfloor$ edge beacons to cover its interior. Note that $\lfloor \frac{e}{21} \rfloor$ edge beacons, represented as the red edges and the red vertices, are sufficient to cover this polyhedron.

Theorem 5 *Let P be an orthogonal polyhedron with e edges. Then $\lfloor \frac{e}{6} \rfloor$ edge beacons are always sufficient to simultaneously cover the interior and exterior of P.*

Proof. Let X, Y, and Z be the number of edges incident to the $\mathcal{X}$-faces, the $\mathcal{Y}$-faces, and the $\mathcal{Z}$-faces respectively. Since $X + Y + Z = 2e$, one of X, Y, or Z is at most $\lfloor \frac{2e}{3} \rfloor$. Let F be the set of edges incident to the $\mathcal{Y}$-faces of P and suppose w.l.o.g. that $|F|$ is at most $\lfloor \frac{2e}{3} \rfloor$. There are two types of faces in F, left and right, and each of them contains four different types of edges: top, bottom, front, and back. Let E_{tb} be the set obtained by selecting first the top edges of the left faces of F and then the bottom edges of the right faces of F. In this manner, we can obtain only four different subsets of edges from F, and thus one of them contains at most $\left\lfloor \frac{|F|}{4} \right\rfloor$ edges of P.

Suppose w.l.o.g. that $|E_{tb}| = \left\lfloor \frac{|F|}{4} \right\rfloor \leq \lfloor \frac{e}{6} \rfloor$. We place a beacon in each $e \in E_{tb}$. Consider the bounding box B of P. Note that the top face of B contains the topmost faces of P, each of which contains at least one element of E_{tb}. Therefore, any point above the top face of B is covered. A similar reasoning can be used to prove that any point to the left of the left face of B, below the bottom face of B, or to the right of the right face of B is covered. A frontmost face of P contains at least the endpoints of two elements of E_{tb}, therefore, any point in front of the front face of B is covered. Analogously, a point to the back of the back face of B is also covered. We only have to prove that any point $p \notin P$ in the interior of the bounding box B can be covered by a beacon placed on an element of E_{tb}.

Let Q_p be the $\mathcal{X}$-plane containing p. Note that Q_p contains one or more polygons produced by the intersection of Q_p with P, and that each bottom vertex of a right edge of a polygon in Q_p and each top vertex of a left edge of a polygon in Q_p is of the form $b \cap Q_p$ for some $b \in E_{tb}$.

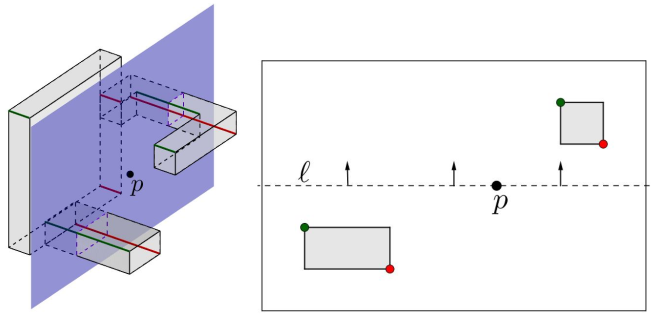
From $p \in Q_p$, shoot two vertical rays, one to the top and one to the bottom, and two horizontal rays, one to the left and one to the right. Two cases may arise, either a ray hits an edge of a polygon in Q_p, or it does not.

Suppose w.l.o.g that the ray ℓ shot up to the top hits a bottom edge e of a polygon in Q_p. If we slide ℓ to the right three cases may occur:

- We reach the endpoint v of e, and v is a convex vertex. Since v is the bottom vertex of a right edge of a polygon in Q_p, it corresponds to an element of E_{tb} in P.
- We reach the endpoint v of e, and v is a reflex vertex. Vertex v is the top vertex of a left edge of a polygon in Q_p, therefore it corresponds to an element of E_{tb} in P.
- We reach a vertical edge of a polygon in Q_p, which corresponds to a left face with an element of E_{tb} in P.

In any case, p is covered by a beacon. The proof for the other cases is similar. Now suppose that none of the rays shot up from p hits an edge of Q_p. Let ℓ be the line parallel to the $\mathcal{X}$-axis that contains p and suppose w.l.o.g that there exists a polygon above p in Q_p. Continuously move ℓ to the top maintaining it horizontal until it hits an edge a of a polygon in Q_p. Since a is a bottom edge, it has a right vertex corresponding to an element of E_{tb}. Figure 6 shows an example of this case. The proof for the case when there exists a polygon below p in Q_p is symmetric.

It follows that the exterior of P is covered. Notice that E_{tb} is composed by the edges selected by the *top-left* and the *bottom-right F-E* rules, it follows from the construction of the proof of Theorem 3 that the interior of P is also covered. □



(a) (b)

Figure 6: (a) An orthogonal polyhedron and a point p in its exterior, p is intersected by an $\mathcal{X}$-plane. (b) The intersection of the $\mathcal{X}$-plane with p. Red vertices represent bottom right edges. Green vertices represent top left edges.

5 Conclusions

In this paper we define a model of attraction by edge beacons. We shown an orthogonal polyhedron whose exterior cannot be covered with vertex beacons. We proved that $\lfloor \frac{e}{12} \rfloor$ edge beacons are always sufficient and $\lfloor \frac{e}{21} \rfloor$ edge beacons are sometimes necessary to cover the interior of an orthogonal polyhedron. We are also interested in covering both the interior and exterior of an orthogonal polyhedron at the same time. We proved that $\lfloor \frac{e}{6} \rfloor$ edge beacons are always sufficient to simultaneously cover the interior and exterior of an orthogonal polyhedron.

Some interesting open problems are: Closing the gap between the upper and lower bounds in both the interior and in the interior-exterior beacon coverage problems in orthogonal polyhedra. Perhaps more challenging is the study of the beacon coverage problem in general polyhedra.

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