Covering orthotrees with guards and beacons

I. Aldana-Galván*[1], J.L. Álvarez-Rebollar†[2], J.C. Catana-Salazar‡[1], N. Marín-Nevárez§[1], E. Solís-Villarreal¶[1], J. Urrutia‖[3], and C. Velarde**[4]

[1]Posgrado en Ciencia e Ingeniería de la Computación, Universidad Nacional Autónoma de México
[2]Posgrado en Ciencias Matemáticas, Universidad Nacional Autónoma de México
[3]Instituto de Matemáticas, Universidad Nacional Autónoma de México
[4]Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México

Introduction

In this paper we study a variant of The Art Gallery Problem in which beacons are used as guards.

Let P be an orthogonal polyhedron. Two points p and q in P are visible to each other if the line segment $\overline{pq}$ is contained in P. A set G of points of P *guards* P if every point $p \in P$ is visible from a point $g \in G$.

A *beacon* is a fixed point in P that can induce a magnetic pull toward itself over all points in P. When a beacon b is activated, points in P move greedily to decrease their Euclidean distance to b. A point p can move along any obstacles it hits on its way towards a beacon b as long as its distance to b keeps monotonically decreasing. Thus, the path from the initial position of p towards a beacon b may alternate between moving in straight line segments contained in the interior of P and line segments on the faces of P.

The piecewise linear path created by the movement of p under the attraction of b is called the *attraction path* of p with respect to b. If the attraction path of p ends in b, it is said that p is *covered* by b. If p reaches a position where it is unable to move in such a way that its distance to b decreases, p is said to be *stuck* and it has reached a local minimum on ∂P, see Figure 1. It is said that a set B of beacons covers P if every point of P is covered by at least one element of B.

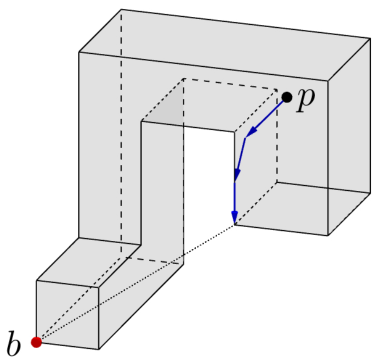


Figure 1: The point p reaches a local minimum when attracted by the beacon b.

Beacon attraction was recently introduced by Biro et al. [1, 2, 3]. This model extends the classical notion of visibility; any object visible from a beacon can be attracted by it following a straight line.

In this paper we prove that $\left\lfloor \frac{n}{8} \right\rfloor$ guards placed on the vertices are always sufficient and sometimes necessary to guard an orthotree having n vertices. We also show that $\left\lfloor \frac{n}{12} \right\rfloor$ beacons placed on the vertices are always sufficient and sometimes necessary to cover a well-separated orthotree having n vertices.

1 Preliminaries

A *polyhedron* in $\mathbb{R}^3$ is a compact connected set bounded by a piecewise linear 2-manifold. A *face* of a polyhedron is a maximal planar subset of its boundary whose interior is connected and non-empty. A polyhedron is *orthogonal* if all of its faces are parallel to the $\mathcal{XY}$, $\mathcal{XZ}$ or $\mathcal{YZ}$ planes. An *edge* is a minimal positive-length straight line segment shared by two faces and joining two vertices of the polyhedron. Each edge, with its two adjacent faces, determines a dihedral angle, internal to the polyhedron. Each such angle is 90 degrees (at a *convex edge*) or 270 degrees (at a *reflex edge*) in an orthogonal polyhedron.

An $\mathcal{X}$*-plane* is a plane that is perpendicular to the

*Email: ialdana@ciencias.unam.mx. Research supported by PAEP from Universidad Nacional Autónoma de México
†Email: chepomich1306@gmail.com. Research supported by PAEP from Universidad Nacional Autónoma de México
‡Email: j.catanas@uxmcc2.iimas.unam.mx. Research supported by PAEP from Universidad Nacional Autónoma de México
§Email: mnjn16@uxmcc2.iimas.unam.mx. Research supported by PAEP from Universidad Nacional Autónoma de México
¶Email: solis_e@uxmcc2.iimas.unam.mx. Research supported by PAEP from Universidad Nacional Autónoma de México
‖Email: urrutia@matem.unam.mx. Research supported by PAPIIT IN102117 from Universidad Nacional Autónoma de México
**Email: velarde@unam.mx

$\mathcal{X}$-axis; we define a *$\mathcal{Y}$-plane* and a *$\mathcal{Z}$-plane* in a similar way. An *$\mathcal{X}$-face* is a face of a polyhedron that is contained in an *$\mathcal{X}$-plane*; we define a *$\mathcal{Y}$-face* and a *$\mathcal{Z}$-face* in a similar way.

A $\mathcal{Y}$-face f of an orthogonal polyhedron P is a *left face* (*right face*), if for any interior point $q \in f$ there is an $\varepsilon > 0$ such that any point at distance less than or equal to ε from q and to the right (left) of f belongs to the interior of P. In a similar way, $\mathcal{Z}$-faces are classified into *top* or *bottom* faces, and $\mathcal{X}$-faces, as *front* or *back* faces.

A *connected polyhedron* P is a *lifting polyhedron* if there exists a *$\mathcal{Z}$-plane* Q, such that for all planes parallel to Q their intersection with P is either empty, or it is a vertical translation of $P \cap Q$.

The next definitions were given by Damian et al. in [5]. An *orthotree* P is an orthogonal polyhedron made of axis-aligned cuboids glued face to face, such that the dual graph of P is a tree. The intersection of two adjacent cuboids in P is 2-dimensional face, namely a non degenerate rectangle. The degree of a cuboid $b_i \in P$ is defined as the degree of its corresponding vertex in the dual graph of P. A *leaf* is a cuboid of degree one; a *connector* is a cuboid b_i of degree two such that its two neighbours are glued to opposite faces of b_i; otherwise, b_i is a *junction*. In this paper we assume that leaves and connectors are only adjacent to junctions, otherwise, we could simply merge adjacent connectors and leaves before applying our proposed solutions. An orthotree is *well-separated* if no junction is adjacent to another junction, that is, all the neighbors of any junction are either leaves or connectors.

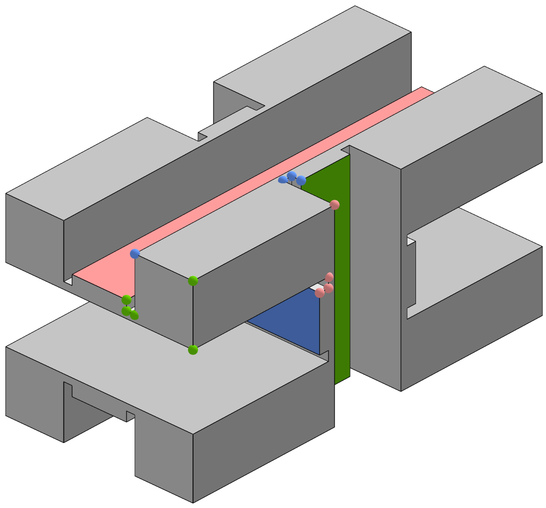
2 Covering orthogonal polyhedra

In this section we show that not all orthogonal polyhedra can be covered with *vertex beacons* (a vertex beacon is a beacon placed on a vertex). This disproves a question posed by J. Urrutia at the 12th Latin American Symposium, Ensenada, Mexico: Is it true that if we place a beacon at every vertex of an orthogonal polyhedron, any point p in the interior of P can choose a beacon b such that b covers p?

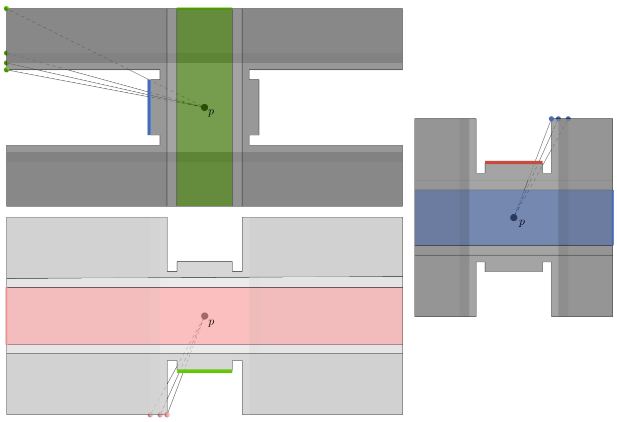
We present now an orthogonal polyhedron P such that if we place a vertex beacon on each vertex of P there is a point p not covered by any of the beacons, see Figure 2a. Our example is based on the octoplex polyhedron, proposed by T. S. Michael in [7]. The octoplex consists of a cuboid with six channels, each one of them going across a different face. It is known that the octoplex cannot be guarded with *vertex guards* (a vertex guard is a guard placed on a vertex). We take a stretched octoplex and attach to the center of each of its channels a slightly narrower cuboid of the same length. This generates a notch in each of the channels of the stretched octoplex, as shown in Figure 2a. We call this polyhedron the *notched octoplex*.

Observe in Figure2 that the attraction path of the "center" point p of P with respect to any of the distinguished vertices leads to a local minimum located at a notch on a channel of P. By symmetry, the same happens with the rest of the vertices of P. Hence, the interior of the notched octoplex cannot be covered with vertex beacons.

Nevertheless, there exist families of orthogonal polyhedra that can be covered (guarded) with vertex beacons (vertex guards), such is the case of the orthotrees. Observe that any point in an orthotree P is visible to at least 8 vertices of P. This observation justifies the study of both covering and guarding orthotrees using vertex beacons and vertex guards, respectively.



(a)



(b)

Figure 2: (a) The notched octoplex, 13 vertices and three faces are colored for reference in Figure 2b. (b) The three different orthogonal projections of P, the line segments show that the 13 vertices of a corner of P (5 blue, 4 green, and 4 rose) cannot attract the center point p, which gets stuck in one of the colored faces; by symmetry the same holds for the rest of the vertices.

3 Guarding orthotrees

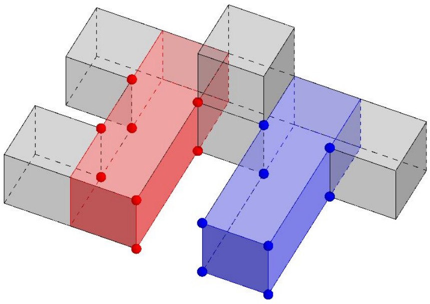


Figure 3: An orthotree with two of its maximal cuboids shown in red and blue.

Theorem 1 *Let P be a orthotree with n vertices. Then $\lfloor \frac{n}{8} \rfloor$ vertex guards are always sufficient and occasionally necessary to guard P.*

Proof. Let P be an orthotree with n vertices. We define a *maximal cuboid* of an orthotree P as a maximal axis-aligned cuboid contained in P (see Figure 3). Note that each maximal cuboid contains at least 8 vertices of P on its boundary. Note also that at most 3 maximal cuboids have a non-empty intersection.

We assign to each maximal cuboid T of P 8 of the vertices of P contained on the boundary of T (see the distinguised vertices in Figure 3).

Let $G = (V, E)$ be the graph where V is the set of vertices assigned to the maximal cuboids of P, and there exists an edge (u, v) in E if u and v are assigned to the same maximal cuboid. Note that, by the choice of vertices for each maximal cuboid, G may be not connected, such is the case for the orthotree shown in Figure 4. The subgraph induced by the vertices assigned to any maximal cuboid of P is a complete graph with 8 vertices.

It is not difficult to see that G is 8-colorable. We 8-color the vertices of G. This partitions the vertices of G into eight chromatic classes, one of which has at most $\lfloor \frac{m}{8} \rfloor$ vertices, where m is the number of vertices of G. Since each maximal cuboid of P has a vertex of each color, it follows that the vertices of any chromatic class guard P. Hence, P can be guarded using at most $\lfloor \frac{n}{8} \rfloor$ of its vertices.

It is easy to see that the three dimensional comb (which is a well-separated orthotree) with k spikes and $n = 8k$ vertices needs at least $k = \frac{n}{8}$ guards to be guarded, see Figure 4. Our result follows. □

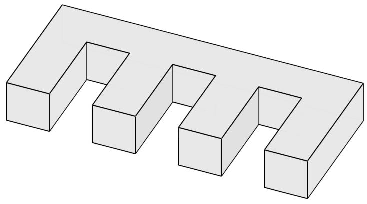


Figure 4: An orthotree that requires $\frac{n}{8}$ vertex guards.

4 Covering well-separated orthotrees

4.1 Upper bound

Theorem 2 *$\lfloor \frac{n}{12} \rfloor$ vertex beacons are always sufficient to cover a well-separated orthotree P having n vertices.*

Proof. Let P be a well-separated orthotree. We construct a graph H whose vertices (*nodes* for clarity) are the junctions and the leaves of P. Two nodes in H are connected by an edge if their corresponding cuboids in P are adjacent or joined by a connector. Observe that H is a tree.

If a node of H represents a leaf of P, we can associate to it four vertices that are not vertices of any other cuboid of P. Each junction j of P is a cuboid such that at least four, and at most eight of its vertices are vertices of P, see Figure 5; associate these vertices to j. Thus each node of H has been assigned at least four vertices of P.

We define the distance between two junctions in P as the distance between the nodes representing them in H. Note that if a junction j is at distance exactly 1 from a junction that contains a beacon b, then both j and the two cuboids adjacent to j (which could be two connectors or a connector and a leaf) are covered by b.

We place beacons in the vertices of P as follows:

Let $v_1, \ldots, v_s$ be a longest path in H. Place a beacon in one of the vertices of the cuboid of P represented by the node v_3 of H. Delete from H the edge (v_3, v_4). Now, consider as H the subtree that contains v_4 and repeat this process until the longest path in H has length less than or equal to 4. An example of this process is shown in Figure 6.

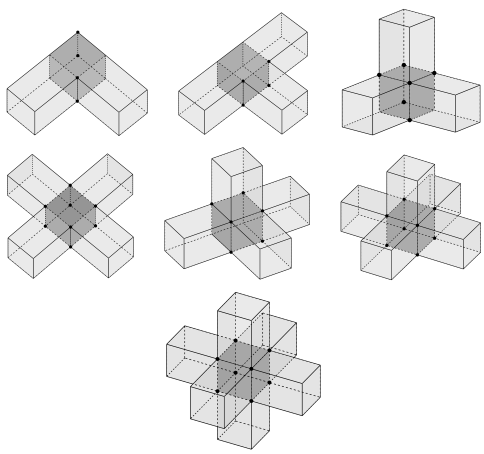


Figure 5: Types of junctions in well-separated orthotrees.

Now we prove that P is covered. We claim that if j is a junction in P, then j is at distance at most 1 from a junction that has a beacon at one of its vertices.

Let j be a junction in P and u its corresponding node in H. Note that u could become a leaf of H as a consequence of a deletion. If u is not a leaf of H before removing its corresponding subtree, then either j contains a beacon or j is joined by a connector to a junction that contains a beacon. Otherwise, in order for u to become a leaf of H, we previously had to cut at least one edge (v, u) incident to u. This implies that we removed a subtree rooted at v, where the junction represented by v has a beacon and is at distance 1 from j, see the node u and its corresponding junction j in Figure 6.

Since a junction j and its two adjacent cuboids are covered by a beacon contained in any other junction at distance 1 from j, and any junction j in P either has a beacon or is at distance 1 from a junction that has a beacon, it follows that any point in P is covered.

We place a beacon for every subtree removed from H. Each removed subtree contains at least three nodes of H. Each node in H represents at least 4 vertices of P. Hence, $\lfloor \frac{n}{12} \rfloor$ beacons placed at the vertices of P are always sufficient to cover P.

□

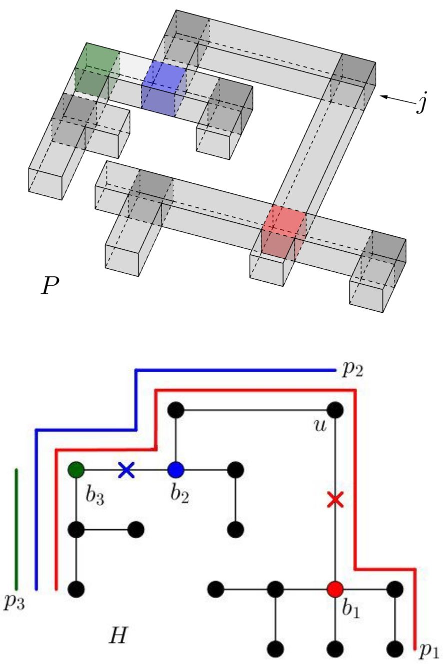


Figure 6: A well-separated orthotree P and the graph H obtained from the junctions (shown in color or dark gray) and leaves of P. In each step, we obtain a longest path in H (shown as p_1, p_2 or p_3, one for each step), then we place a beacon in the junction of P corresponding to the colored node in H (which is shown with the same color as the path), and finally we separate the subtrees by deleting the corresponding edge (marked by a cross of the same color).

The previous proof does not apply to orthotrees that are not well-separated. The reason is that, if an orthotree is not well-separated, then some of its junctions are not vertex disjoint. Therefore, it is not true that we can always assign four vertices to the nodes of the tree that we constructed for the proof.

Conjecture 3 $\lfloor \frac{n}{12} \rfloor$ *vertex beacons are always sufficient to cover an orthotree having n vertices.*

4.2 Lower bound

Bae et al. [4] proved that $\lfloor \frac{n}{6} \rfloor$ beacons, placed at reflex vertices, are always sufficient and sometimes necessary to cover an orthogonal polygon with n vertices. For the lower bound, they gave the construction of an orthogonal spiral P_r with r reflex vertices which needs $\lceil \frac{r}{3} \rceil = \lfloor \frac{n}{6} \rfloor$ beacons in order to be covered.

To obtain the lower bound in $\mathbb{R}^3$ we can construct a lifting polyhedron P from the orthogonal spiral polygon given by Bae et al. [4]. This orthogonal spiral polyhedron is a well-separated orthotree and has twice as many vertices as the orthogonal spiral polygon. Therefore, it follows that $\lfloor \frac{n}{12} \rfloor$ vertices are necessary to cover P.

We are grateful to the anonymous referees for their helpful suggestions.

References

[1] M. Biro, Beacon-based routing and guarding, *PhD thesis*, State University of New York at Stony Brook (2013).

[2] M. Biro, J. Gao, J. Iwerks, I. Kostitsyna, and J. Mitchell, Beacon-based routing and coverage, In *21st Fall Workshop on Computational Geometry* (2011).

[3] M. Biro, J. Iwerks, I. Kostitsyna, and J. S. Mitchell, Beacon-based algorithms for geometric routing, In *Workshop on Algorithms and Data Structures, Springer* (2013), 158–169.

[4] S. W. Bae, C.-S. Shin, and A. Vigneron, Tight bounds for beacon-based coverage in simple rectilinear polygons, In *Latin American Symposium on Theoretical Informatics, Springer*, (2016), 110–122.

[5] M. Damian, R. Flatland, H. Meijer, and J. O'Rourke, Unfolding well-separated orthotrees, In *15th Annu. Fall Workshop Comput. Geom, Citeseer* (2005), 23–25.

[6] J. Kahn, M. Klawe, and D. Kleitman, Traditional galleries require fewer watchmen, *SIAM Journal on Algebraic Discrete Methods* **4(2)** (1983), 194–206.

[7] T. Michael, Guards, galleries, fortresses, and the octoplex, *The College Mathematics Journal* **42(3)** (2011), 191–200.