

# Conservation principles and action schemes in the synthesis of geometric concepts

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## Diagrammatic reasoning

- Reasoning
- Learning
- Perception
- Design and creativity
- Theorem proving
- Ubiquitous in science and engineering

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## Diagrammatic reasoning

- How diagrammatic knowledge is represented
- What kind of inferences are supported by diagrams
- How external representations participate in this process

**This is a problem in knowledge representation!**

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## Some general questions about diagrams

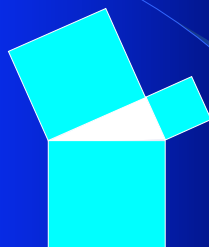
- What is their expressive power
- Why can they be interpreted so effectively
- What is the relation between logic and diagrammatic reasoning

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## Theorem of Pythagoras



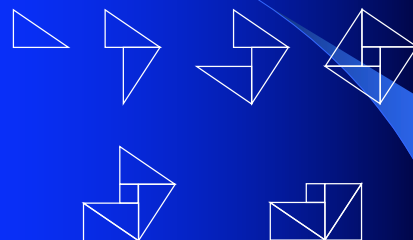
$$h^2 = a^2 + b^2$$

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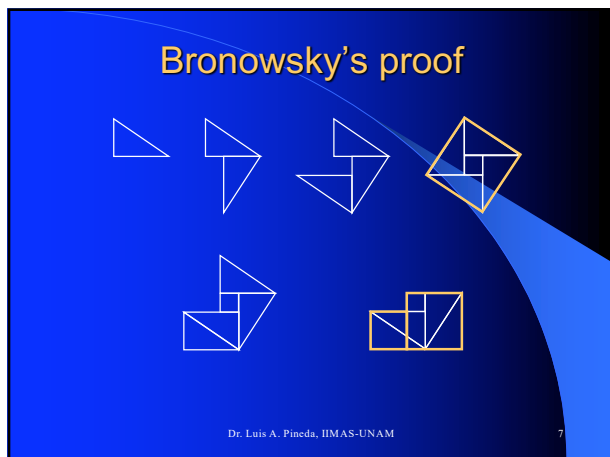
## Bronowsky's proof



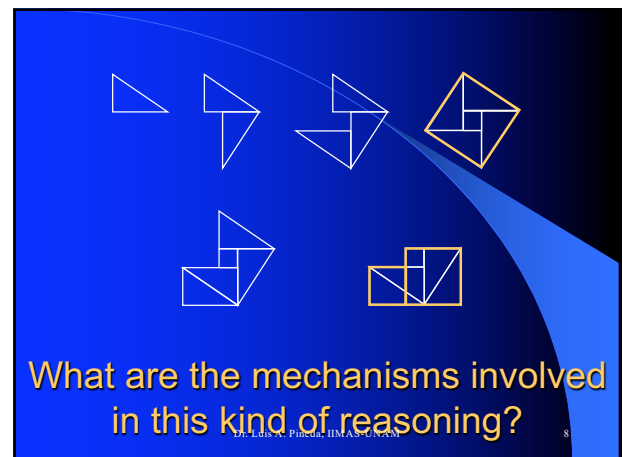
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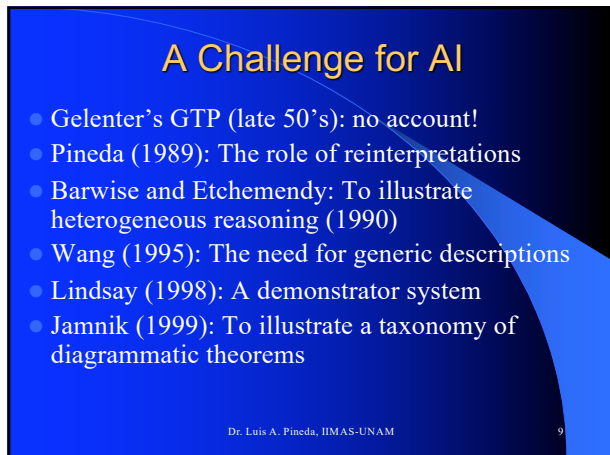
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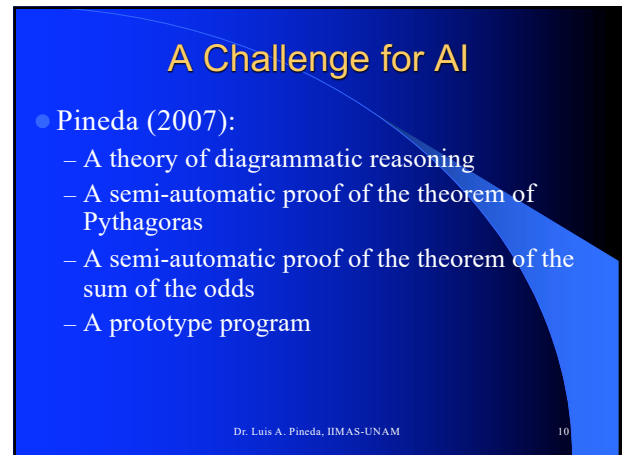
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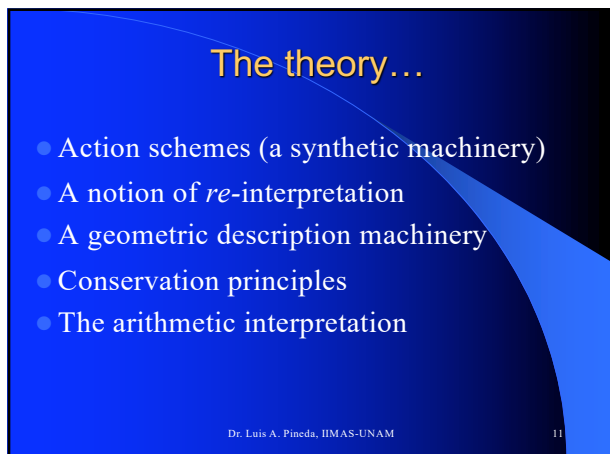
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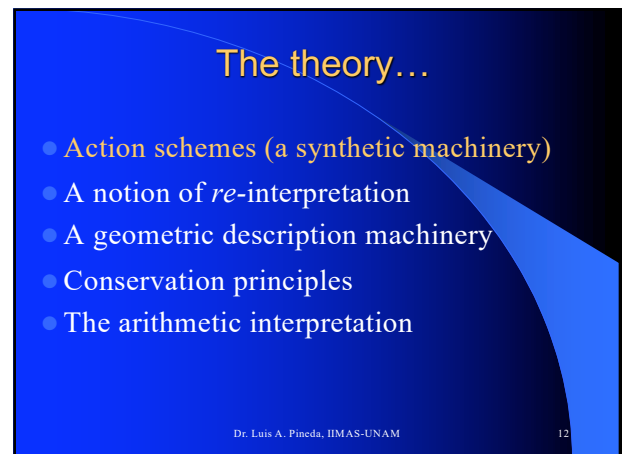
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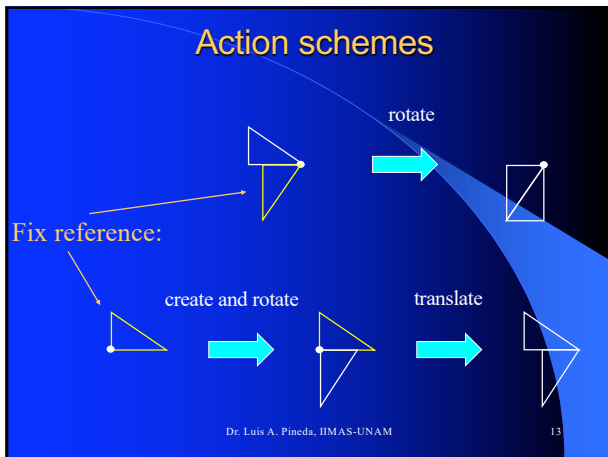
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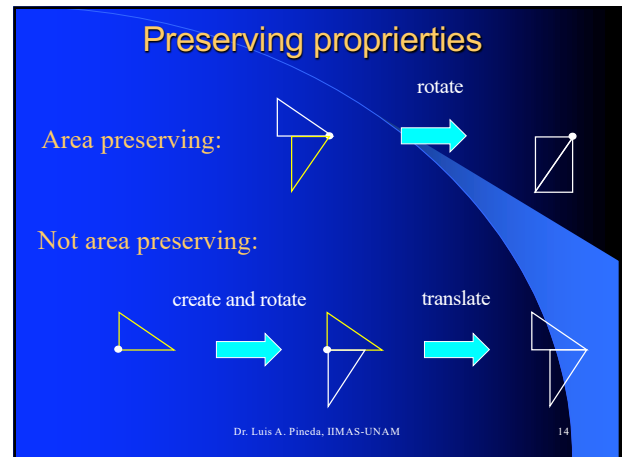
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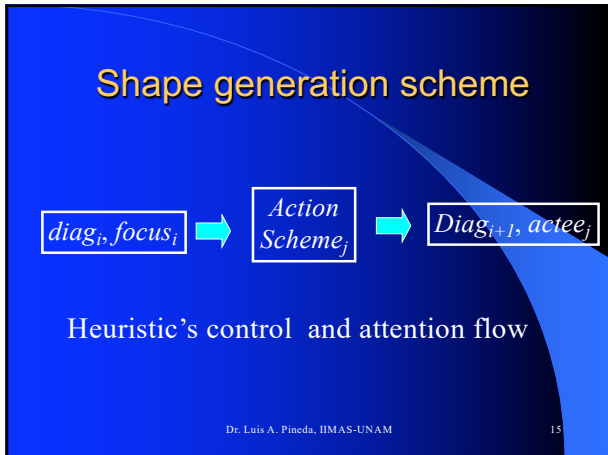
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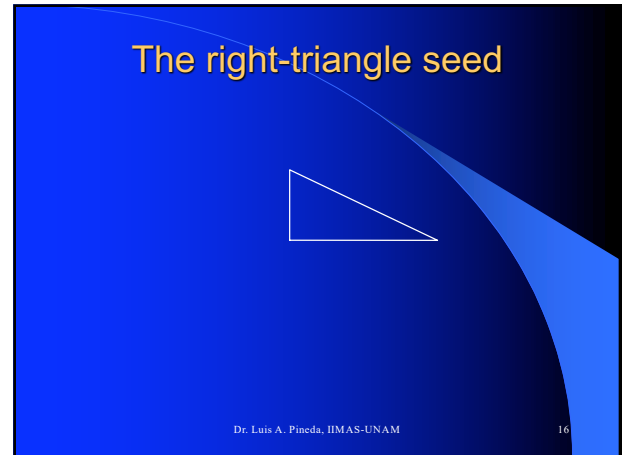
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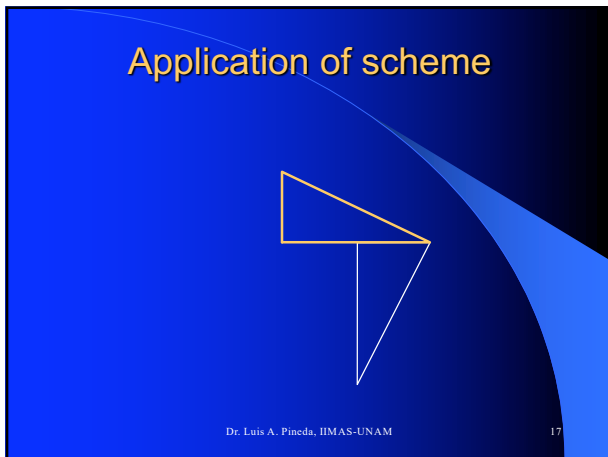
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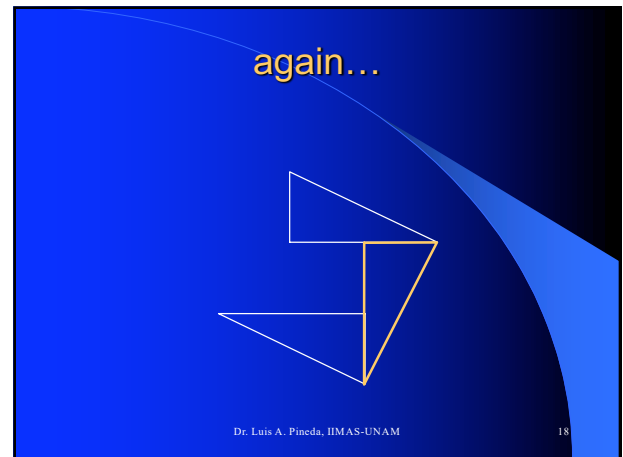
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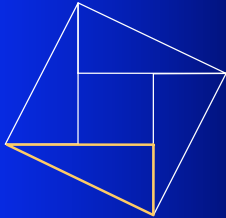


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... and again!

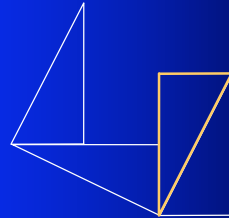


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First area preserving transformation...

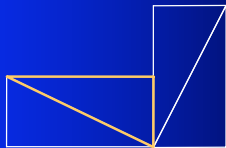


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Second area preserving transformation



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The theory...

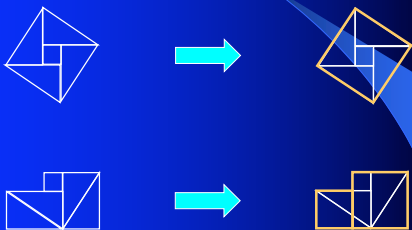
- Action schemes (a synthetic machinery)
- A notion of *re*-interpretation
- A geometric description machinery
- Conservation principles
- The arithmetic interpretation

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The *re*-interpretations and “emerging” objects

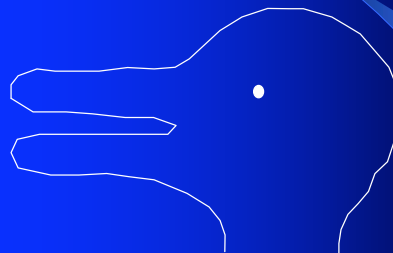


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A change in the conceptual perspective!



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## A problem of description...



## A perceptual inference?

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## We need the relevant description



*A square on the hypotenuse of a right-triangle*



*The union of a square on a right side of a right triangle and a square on the other right side of the same right triangle*

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## The theory...

- Action schemes (a synthetic machinery)
- A notion of *re*-interpretation
- A geometric description machinery
- Conservation principles
- The arithmetic interpretation

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## Concepts

- Concepts (i.e. knowledge objects) can be represented in computers
- Turing Machines compute functions
- So, concepts are represented through functions
- The challenge is to find such functions
- In the present case, the functions representing geometric and arithmetic concepts that are expressed through diagrams!

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## Geometric description machinery

- A geometric signature to refer to geometric objects, properties and relations
- The functional abstractor operator to express geometric concepts
- A geometric descriptor operator to refer to (contextually dependent) emerging objects:

$$T \leq f$$

– If  $f(A)$  is true  $(T \leq f) = T$  where  $T$  is a term of any geometric sort which contains (possible) variables in  $f$

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## Generic description

- Diagram:



- Description:  $y \leq f_1$

- where:

$$f_1 = \lambda x \lambda y. \text{right\_triang}(x) \ \& \ \text{square}(y) \ \& \ \text{side}(\text{hipotenuse}(x), y)$$

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## Generic description

- Diagram:



- Description:

$union(y, z) \leq f_2$

where:

$f_2 = \lambda x \lambda y \lambda z. right\_triang(x) \ \& \ square(y) \ \& \ square(z) \ \& \ side(side\_a(x), y) \ \& \ side(side\_b(x), z)$

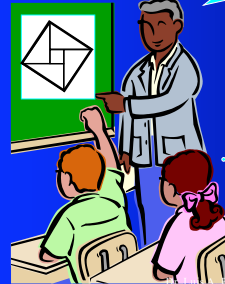
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## Diagrams and descriptions

A square on the hypotenuse of a right-triangle



$y \leq f_1$

Descriptions as internal Representations?

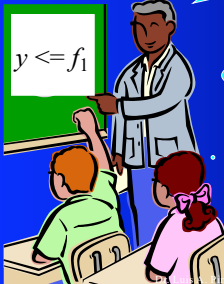
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## Diagrams and descriptions

A square on the hypotenuse of a right-triangle



$y \leq f_1$

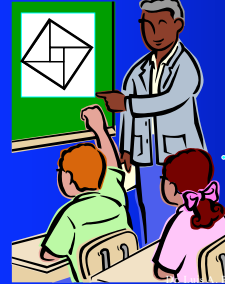
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## Diagrams and descriptions

$y \leq f_1$



$y \leq f_1$

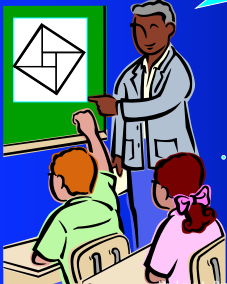
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## Functions represent meanings!

A square on the hypotenuse of a right-triangle



Meaning

$y \leq f_1$  represents a generic concept!

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## The theory...

- Action schemes (a synthetic machinery)
- A notion of *re*-interpretation
- A geometric description machinery
- Conservation principles
- The arithmetic interpretation

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We need to state a property holds for different diagrams...



*Area of a square on the hypotenuse of a right-triangle*



*Area of the union of a square on a right side of a right triangle and a square on the other right side of the same right triangle*

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This is a relation between generic descriptions...



$area(y \leq f_1)$



$area(union(y, z) \leq f_2)$

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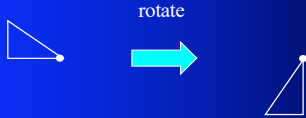
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## Conservation principles

- Generalized concept of equality for geometrical properties
- Global principle of conservation of area:

$$\lambda P \lambda Q (area(P) = area(Q))$$



- The application of the principle is granted if the action scheme producing the transformation preserves the conservation property

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## Conservation principles

- Structured principle of conservation of area:

$$\lambda P \lambda Q \lambda x (area(P(x)) = area(Q(x)))$$

- $P$  and  $Q$  are generic descriptions of geometrical objects or configurations
- $x$  is a generic reference object

- An interpretation act (under the appropriate conditions) is represented by a functional application operation!

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## Synthesis of geometric concepts



$$\lambda P \lambda Q \lambda x (area(P(x)) = area(Q(x)))(y \leq f_1)$$

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## Synthesis of geometric concepts



$$\lambda Q \lambda x (area(y \leq f_1(x)) = area(Q(x)))$$

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## Synthesis of geometric concepts



$$\lambda Q \lambda x (area(y \leq f_1(x)) = area(Q(x))(union(y, z) \leq f_2))$$

The application is permitted if the the diagram is modified by an area preserving (sequence of) transformation

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## Synthesis of geometric concepts



$$\lambda x (area(y \leq f_1(x)) = area(union(y, z) \leq f_2(x)))$$

The function representing the geometric concept of the Theorem of Pythagoras!

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## The geometric concept

$$f_{TP} = \lambda x (area(y \leq f_1(x)) = area(union(y, z) \leq f_2(x)))$$

Where:

$$f_1 = \lambda x \lambda y. right\_triang(x) \& square(y) \& side(hipotenuse(x), y)$$

$$f_2 = \lambda x \lambda y \lambda z. right\_triang(x) \& square(y) \& square(z) \& side(side\_a(x), y) \& side(side\_b(x), z)$$

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## The extension of the concept...

Are these in the Pythagorean relation?



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## The extension of the concept...

Are these?



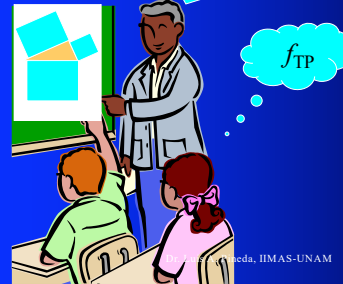
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## Representation of meanings!

The area on the hypotenuse of a right triangle is the same as the area of the union of the squares on its right sides



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## The theory...

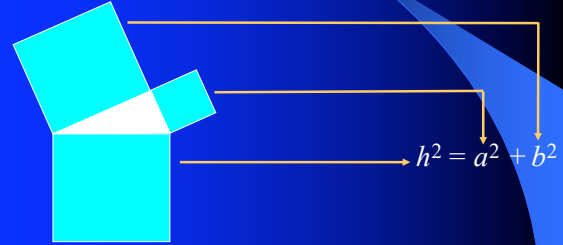
- Action schemes (a synthetic machinery)
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## A mapping from the geometry into the arithmetic



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## The representation function

$\phi(x \leq f) = \lambda u. u^2$  if the type of  $x$  in  $f$  is  $sq$

$\phi(union) = +$

$\phi(g(y_1, y_2) \leq f) = \phi(g)(\phi(y_1 \leq f), \phi(y_2 \leq f))$

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## The mapping

Diagram:



The arithmetic concept:

$$\phi(y \leq f_1) = \lambda u. u^2$$

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## The mapping...

The diagram:



The arithmetic concept:

$$\phi(union(y, z) \leq f_2) = \lambda v. v^2 + \lambda w. w^2$$

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## The mapping

- The geometric principle:
  - $\lambda P \lambda Q \lambda x (area(P(x)) = area(Q(x)))$
- The arithmetic principle:
  - $\lambda P \lambda Q (P = Q)$
  - Concept of global arithmetic equality!

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## Diagrammatic Derivations

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## A three-tier tandem process

- The synthesis of geometric form
- The synthesis of the geometric concept
- The synthesis of the arithmetic concept

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## The seed...

- Diagram:



- Principle of conservation of area:
  - $\lambda P \lambda Q \lambda x (area(P(x)) = area(Q(x)))$
- Concept of the global arithmetic equality:
  - $\lambda P \lambda Q (P = Q)$

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## Synthesis of form

- Diagram:



- Principle of conservation of area:
  - $\lambda P \lambda Q \lambda x (area(P(x)) = area(Q(x)))$
- Concept of the global arithmetic equality:
  - $\lambda P \lambda Q (P = Q)$

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## Synthesis of form

- Diagram:



- Principle of conservation of area:
  - $\lambda P \lambda Q \lambda x (area(P(x)) = area(Q(x)))$
- Concept of the global arithmetic equality:
  - $\lambda P \lambda Q (P = Q)$

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## Synthesis of form

- Diagram:



- Principle of conservation of area:
  - $\lambda P \lambda Q \lambda x (area(P(x)) = area(Q(x)))$
- Concept of the global arithmetic equality:
  - $\lambda P \lambda Q (P = Q)$

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## Synthesis of form

- Diagram:



- Principle of conservation of area:  
–  $\lambda P \lambda Q \lambda x (area(P(x)) = area(Q(x)))$
- Concept of the global arithmetic equality:  
–  $\lambda P \lambda Q (P = Q)$

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## First reinterpretation

- Reinterpretations preserve area:



- Concepts construction:  
–  $\lambda P \lambda Q \lambda x (area(P(x)) = area(Q(x)) (w \leq f_1))$   
–  $\lambda P \lambda Q (P = Q) (\lambda u. u^2)$

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## First reinterpretation

- Reinterpretation:



- Concepts construction:  
–  $\lambda Q \lambda x (area(w \leq f_1(x)) = area(Q(x)))$   
–  $\lambda Q (\lambda u. u^2 = Q)$

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## Synthesis of form

- Diagram:



- Concepts construction:  
–  $\lambda Q \lambda x (area(w \leq f_1(x)) = area(Q(x)))$   
–  $\lambda Q (\lambda u. u^2 = Q)$

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## Synthesis of form

- Diagram:



- Concepts construction:  
–  $\lambda Q \lambda x (area(w \leq f_1(x)) = area(Q(x)))$   
–  $\lambda Q (\lambda u. u^2 = Q)$

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## Second reinterpretation

- Reinterpretation:



- Concepts construction:  
–  $\lambda Q \lambda x (area(w \leq f_1(x)) = area(Q(x)) (union(y, z) \leq f_2))$   
–  $\lambda Q (\lambda u. u^2 = Q) ((\lambda v. v^2, \lambda w. w^2))$

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## Second reinterpretation

- Reinterpretation:



- Concepts construction:

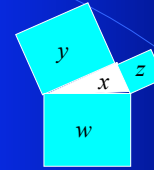
- $\lambda x(\text{area}(w \leq f_1(x)) = \text{area}(\text{union}(y, z) \leq f_2(x)))$
- $\lambda u.u^2 = +(\lambda v.v^2, \lambda w.w^2)$

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## Program transformation rules



$$\lambda x.\lambda w.\lambda y.z.(\text{area}((w \leq f_1)(x, w)) = \text{area}((\text{union}(y, z) \leq f_2)(x, (y, z))))$$

$$\lambda u, v, w.u^2 = v^2 + w^2$$

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## Questions about diagrams

- What is their expressive power
- Why can they be interpreted so effectively
- What is the relation between logic and diagrammatic reasoning

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## Questions about diagrams

- What is their expressive power
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## Diagrams and abstraction

- A common view is that diagrams are good for expressing concrete information but...
- There is a limitation in the abstractions that can be expressed
- The theory of graphical specificity (Stenning and Oberlander, 1995)

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## We can have concrete interpretations...

- Diagram:



- Description:

$$sq_1 \leq \text{right-triangle}(rt_1) \ \& \ \text{square}(sq_1) \ \& \ \text{side}(\text{hipotenuse}(rt_1), sq_1)$$

... and deal with the ambiguity!

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We can limit the expressive power of the representational language...

Diagram:



Descripción:

$union(sq_1, sq_2) \Leftarrow right\_triang(rt_1) \ \& \ right\_triang(rt_2) \ \& \ square(sq_1) \ \& \ square(sq_2) \ \& \ side(side\_a(rt_1), sq_1) \ \& \ side(side\_b(rt_2), sq_2)$

and face the limitations of the medium!

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## Representation of meanings

The area on the hypotenuse of a right triangle is the same as the area of the union of the squares on its right sides



Through the *lambda calculus* we represent interpretations of diagrams ...

NOT diagrams!!!<sup>4</sup>

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## Diagrams and abstraction

- The present theory shows that diagrams can be given generic (fully abstract) interpretations!
- A representation is specified through:
  - The external symbols and configurations
  - The interpretation process
  - The language to represent the interpretations does not need to have a limited expressivity (e.g. propositional logic)
- Diagrammatic proofs are genuine proofs!

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## Questions about diagrams

- What is their expressive power
- Why can they be interpreted so effectively
- What is the relation between logic and diagrammatic reasoning

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## Reasoning with concrete representations

- Vision provides concrete interpretations of shapes directly
- Easy... if the problem has a concrete nature!
- Concrete problems can often be expressed through diagrams
- But, if the problem demands abstraction (e.g. an infinite number of instances) concrete resources (memory and computational time) run out very quickly!

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## Abstractions capture change implicitly!

- Two dimensions of change:
  - The parameters of the diagrammatic objects
  - Different diagrammatic configurations that have the same description (i.e. equivalent in relation to the task)

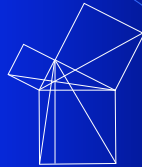
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## Abstractions account for equivalent objects!

- Diagram:



- Description:  $y \leq f_1$

- where:

$$f_1 = \lambda x \lambda y. \text{right\_triang}(x) \ \& \ \text{square}(y) \ \& \ \text{side}(\text{hipotenuse}(x), y)$$

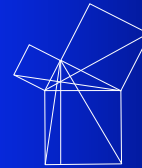
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## Abstractions account for equivalent objects!

- Diagram:



- Description:

$$\text{union}(y, z) \leq f_2$$

- where:

$$f_2 = \lambda x \lambda y \lambda z. \text{right\_triang}(x) \ \& \ \text{square}(y) \ \& \ \text{square}(z) \ \& \ \text{side}(\text{side}_a(x), y) \ \& \ \text{side}(\text{side}_b(x), z)$$

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## Diagrammatic reasoning is monotonic!

- In spite of the change in the geometric form and regardless the values of the parameters of diagrammatic objects, the synthesis of the geometric and arithmetic processes is monotonic

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## Reading a diagrammatic sequence!

- Incremental interpretation:
  - *every man is mortal*
  - $\lambda P \lambda Q \lambda x. (P(x) \supset Q(x))(\text{man})(\text{mortal})$
  - $\lambda Q \lambda x. (\text{man}(x) \supset Q(x))(\text{mortal})$
  - $\lambda x. (\text{man}(x) \supset \text{mortal}(x))$
- There is not a change to account for!
- Natural language quantifiers can be seen as conservation principles!

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## Reasoning with abstractions is easy!

- Abstractions are small finite representational objects (that represent interpretations) that can be used in thought process as units, but have a very large, perhaps infinite, extension

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## What is hard is to produce the relevant abstractions!



*A square on the hypotenuse of a right-triangle*



*The union of a square on a right side of a right triangle and a square on the other right side of the same right triangle*

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## Abstracting on concrete descriptions?



$duck_1 \leq duck(duck_1) ?$

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## Constructing the abstraction directly!



$x \leq \lambda x. duck(x)$

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## Generation of abstract descriptions ...

- The extensional representation
- Visualisations (i.e. Reinterpretations)
- Domain knowledge (e.g. Geometry)
- Knowledge about the aims of the task (e.g. theorem proving and discovery)

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## Perceptual inference



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## Questions about diagrams

- What is their expressive power
- Why can they be interpreted so effectively
- What is the relation between logic and diagrammatic reasoning

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## The axiomatic method



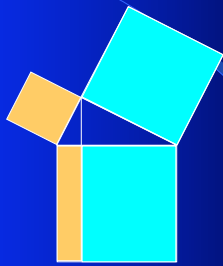
**Proposition 47, Euclid's Elements**

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## The axiomatic method



**A simpler problem!**

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## Reinterpretations

- Enrich the problem-solving space
- Interesting emerging objects belong to the enriched space
- The recognition of emerging objects depends on the interpretation process, but also on the nature of the external representation!
- The process is genuinely synthetic and synthesized objects cannot be found through analysis!

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## The paper:

Luis A. Pineda, Conservation principles and action schemes in the synthesis of geometric concepts, *Artificial Intelligence* 171 (March, 2007) 197-238.

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**Thanks very much!**

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