On the Classification of Resolvable 2-$(12, 6, 5c)$ Designs

Luis B. Morales and Carlos Velarde
IIMAS, Universidad Nacional Autónoma de México
México, DF, 04510, México

Abstract

In this paper we describe a backtrack search over parallel classes with a partial isomorph rejection to classify resolvable 2-$(12, 6, 5c)$ designs. We use the intersection pattern between the parallel classes and the fact that any resolvable 2-$(12, 6, 5c)$ design is also a resolvable 3-$(12, 6, 2c)$ design to effectively guide the search. The method was able to enumerate all nonsimple resolutions and a subfamily of simple resolutions of a 2-$(12, 6, 15)$ design. The method is also used to confirm the computer classification of the resolvable 2-$(12, 6, 5c)$ designs for $c \in \{1, 2\}$. A consistency checking based on the principle of double counting is used to verify the computation results.

Keywords: *parallel class; resolvable t-design; backtrack search with isomorph rejection; pruning*

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1 Introduction

A *design* is a pair $(V, \mathcal{D})$ where V is a v-set of *points* and $\mathcal{D}$ is a collection of subsets of V called *blocks*. A t-(v, k, λ) design is a design $(V, \mathcal{D})$ such that each block is of size k and each t-subset of V is contained in exactly λ blocks. It is easy to see that each point occurs in the same number r of blocks. Similarly, if b denotes the number of blocks, we have that

$$vr = bk, \qquad r\binom{k-1}{t-1} = \lambda\binom{v-1}{t-1}. \tag{1}$$

A design with $t = 2$ is called a *balanced incomplete block design* (BIBD). For such designs, the second equation of (1) reduces to $r(k-1) = \lambda(v-1)$. A design is called *simple* if it contains no repeated blocks.

Two designs are *isomorphic* if there exists a bijection between the point sets that maps blocks onto blocks; such a bijection is called an *isomorphism*. An *automorphism* of a design is an isomorphism of the design onto itself. The *(full) automorphism group* of a design consists of all of its automorphisms with function composition as the group operation.

A *parallel class* in a design is a set of blocks that partition the point set. A *resolution* of a design is a partition of the blocks into parallel classes. A design is called *resolvable* if it has a resolution. Two resolutions are isomorphic if there exists an isomorphism between the designs mapping each parallel class of the first resolution into a parallel class of the second. When $v = 2k$ a resolvable design has a unique resolution.

Classification of designs constitutes a significant field in design theory. In particular, there are many papers dealing with the existence or classification of resolvable designs with given parameters, see for example [3], [4], [6], [9] and [10]. A comprehensive and detailed study of approaches for constructing and classifying designs is contained in [6]. Although classification algorithms for designs have been improved along the years together with increasing speed of computers there exist several open problems in this area. In particular, the number of pairwise nonisomorphic resolvable 2-$(12, 6, 15)$-BIBDs remains unknown. In [14], $225,970$ nonisomorphic resolvable 2-$(12, 6, 15)$-BIBDs were constructed.

In this paper we describe a backtrack algorithm over parallel classes with a partial isomorph rejection to classify resolvable 2-$(12, 6, 5c)$ designs. Our algorithm was able to classify, up to isomorphism, all nonsimple resolutions as well as a subfamily of simple resolutions of a 2-$(12, 6, 15)$ design. The algorithm was, however, unable to enumerate the remaining resolvable designs because there are too many such designs to find, given existing computational resources. Also, our algorithm was able to classify the resolvable 2-$(12, 6, 5c)$ designs for $c = 1$ and $c = 2$. A total of one, and 545 nonisomorphic designs were found for $c = 1$ and $c = 2$, respectively. Our results agree with the results reported in [6], for $c = 2$. Thus, this paper provides an independent verification of the computer classification for the resolvable 2-$(12, 6, 10)$ designs. Furthermore, our program confirms the results obtained in [14] for orthogonal resolutions of a 2-$(12, 6, 15)$ design. Finally, a consistency checking based on the basic principle of double counting is also used to verify our results.

Our enumeration process has two main stages. In the first we generate a set of collections of parallel classes (the initial structures). The second stage consists in determining all extensions of each of these initial structures into a resolution of a 2-$(12, 6, 5c)$ design.

The following Theorem of Alltop [1] will play a central role in our search algorithm.

Theorem 1 *A resolvable t-$(2k, k, \lambda)$ design with t even is a resolvable $t+1$-$(2k, k, \lambda')$ design with $\lambda' = \lambda(k-t)/(2k-t)$ and vice versa.*

In our search for resolutions of BIBDs, we use intersection patterns between parallel classes as well as Theorem 1 in order to produce initial structures, which guide the search. Using Theorem 1, we find relationships between pairs and triples of points of the design. These relationships are used to effectively prune the search tree. Our isomorph rejection is based on the graph canonical labeling software Nauty [12].

In Section 2 we prove some results that restrict the structure of any resolution of a 2-$(12, 6, 5c)$ design. Section 3 describes the backtracking algorithm over parallel classes to construct initial structures for the search of all resolutions of such designs. Section 4 outlines the backtrack search to extend an initial structure into resolutions of a 2-$(12, 6, 5c)$ design. In Section 5 we describe a new idea to prune the search tree based on relationships between pairs and triples of points of the design. The computations showed that there are $58, 619, 818, 970$ nonsimple, and at least $30, 885, 758, 702$ simple nonisomorphic resolvable 2-$(12, 6, 15)$ designs. Finally, in order to gain confidence in the correctness of the classification, we perform a consistency check based on double counting.

2 Preliminaries

In this section we give some results that restrict the structure of the resolutions of a 2-$(12, 6, 5c)$ design. For these resolutions $b = 22c$, $r = 11c$ and, by Theorem 1, each triple of points of the design occurs $2c$ times.

Let $R_x = \{D_{x,1}, D_{x,2}\}$ and $R_y = \{D_{y,1}, D_{y,2}\}$ be two parallel classes in a resolution of a 2-$(12, 6, 5c)$ design. Define their *parallel class intersection* matrix (PCIM) as the 2×2 matrix $M(x, y) = (m_{ij}(x, y))$, where $m_{ij}(x, y) = |D_{x,i} \cap D_{y,j}|$ (see, for example, [4, 9, 10]). Since each point belongs to exactly one block of a parallel class, the column and row sums in any PCIM is 6. Then, up to permutation of rows and columns, for resolvable 2-$(12, 6, 5c)$ designs the only possible PCIMs are

$$\mathcal{T}_1 = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}, \quad \mathcal{T}_2 = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}, \quad \mathcal{T}_3 = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}, \quad \mathcal{T}_4 = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}.$$

For any parallel class R_x in a resolution $\mathcal{R}$ of a 2-$(12, 6, 5c)$ design, we denote by n_i the number of parallel classes R_y in $\mathcal{R}$ such that the PCIM $M(x, y)$ is $\mathcal{T}_i$, for $i = 1, \ldots, 4$. Since $\mathcal{R}$ has $11c$ parallel classes, for the parallel class

R_x, there are $11c-1$ PCIMs with respect to the other parallel classes. It follows by counting pairs of points occurring in a block of R_x that

$$\begin{aligned} n_1+n_2+n_3+n_4 &= 11c-1 \\ 15n_1+10n_2+7n_3+6n_4 &= (5c-1)\binom{6}{2}. \end{aligned} \tag{2}$$

Any solution in nonnegative integers (n_1, n_2, n_3, n_4) of system (2) is called an *intersection pattern* of R_x in $\mathcal{R}$. The set of all possible nonnegative integer solutions of (2) is denoted by $S(c)$. Solving the equation system we get $|S(1)| = 1$, $|S(2)| = 4$ and $|S(3)| = 9$. The sets $S(2)$ and $S(3)$ are shown in Table 1.

Table 1: Possible intersection patterns of any parallel class in a resolution. (a) For 2-$(12, 6, 10)$ designs; (b) for 2-$(12, 6, 15)$ designs.

(a)

	n_1	n_2	n_3	n_4
1	1	0	0	20
2	0	2	1	18
3	0	1	5	15
4	0	0	9	12

(b)

	n_1	n_2	n_3	n_4
1	2	0	0	30
2	1	2	1	28
3	1	1	5	25
4	1	0	9	22
5	0	4	2	26
6	0	3	6	23
7	0	2	10	20
8	0	1	14	17
9	0	0	18	14

Now the set $S(c)$ is used to divide the set of resolutions of a 2-$(12, 6, 5c)$ design into $|S(c)|$ nonisomorphic classes. For this purpose we impose a total ordering on $S(c)$. Although any ordering can be used, we preferred to use the reverse lexicographic ordering (see Table 1). Clearly $(c-1, 0, 0, 10c)$ and $(0, 0, 9c-9, 2c+8)$ are respectively the first and the last elements of $S(c)$ for any $c \geq 1$. A parallel class in a resolution $\mathcal{R}$ is called of *type* s if its intersection pattern in $\mathcal{R}$ is the sth element of $S(c)$. A resolution $\mathcal{R}$ is called of *type* s if it has at least one parallel class of type s, but it has no parallel class of type $i < s$. Note that two resolutions of different types cannot be isomorphic.

Let $\mathcal{R}$ be a resolution of a 2-$(12, 6, 5c)$ design. For any parallel class R_x in $\mathcal{R}$, we define

$$F(R_x) = \{R_y \in \mathcal{R} \mid M(x, y) \neq \mathcal{T}_4\}.$$

If R_x is a parallel class of type s in a resolution $\mathcal{R}$ of type s, then the collection of parallel classes $F(R_x)$ is called an *initial structure of type s* of $\mathcal{R}$.

Lemma 2 *Let* $R_x = (B_{x1}, B_{x2})$ *be a parallel class of a resolvable* 2-$(12, 6, 5c)$ *design. Let* T *be a 3-subset of* B_{xj} *for either* $j = 1$ *or* $j = 2$, *and* $T^c = B_{xj} - T$. *Then both* T *and* T^c *have the same number of occurrences in* $F(R_x)$.

Proof. Let $R_y \in \mathcal{R} - F(R_x)$. Thus $M(x, y) = \mathcal{T}_4$. It follows that T occurs in one of the blocks of R_y if and only if T^c occurs in the other block of R_y. Then, T and T^c occur the same number of times in $\mathcal{R} - F(R_x)$. Also, from Theorem 1, both T and T^c occur $2c$ times in the design. Therefore, both T and T^c occur the same number of times in $F(R_x)$. □

3 Initial structures

In this section we describe the backtracking over parallel classes used for constructing all nonisomorphic initial structures of all resolutions of a 2-$(12, 6, 5c)$ design.

We fix some notation: Let $X = \{0, \dots, 5\}$ and $Y = \{6, \dots, 11\}$. Put $V = X \cup Y$. Let $\mathcal{C}$ be the set of all pairs (C_1, C_2) such that $\{C_1, C_2\}$ is a partition of V with $|C_1| = |C_2| = 6$. For each partition $\{A, B\}$ of X, with $A = \{x_1, x_2, x_3\}$ and $B = \{x_4, x_5, x_6\}$, define the set

$$\mathcal{P}(A, B) = \{(A \cup \{y_1, y_2, y_3\}, B \cup \{y_3, y_4, y_6\}) \in \mathcal{C} \mid \\ \{\{y_1, y_2, y_3\}, \{y_3, y_4, y_6\}\} \text{ is a partition of } Y\}.$$

It follows that $|\mathcal{P}(A, B)| = \binom{6}{3}$.

Let $\mathcal{R} = (R_1, \cdots, R_r)$ be a resolution of type s of a 2-$(12, 6, 5c)$ design. Suppose that R_1 is a parallel class of type s in $\mathcal{R}$. Let (n_1, n_2, n_3, n_4) be the intersection pattern of the parallel class R_1. Without loss of generality we can assume that $R_i = (X, Y)$, for $i = 1, \dots, n_1 + 1$. Also, without loss of generality we may assume that $R_{n_1+2} = (Z, V - Z)$, where $Z = \{0, 1, 2, 3, 4, 6\}$ if $n_2 > 0$, or $Z = \{0, 1, 2, 3, 5, 6\}$ if $n_2 = 0$ and $n_3 > 0$. Note that $M(1, n_1 + 2) = \mathcal{T}_2$ if $n_2 > 0$ or $M(1, n_1 + 2) = \mathcal{T}_3$ if $n_2 = 0$ and $n_3 > 0$.

In order to generate all possible initial structures for the resolutions of type s of a 2-$(12, 6, 5c)$ design, we use a backtracking algorithm over parallel classes. The tuple $(R_1, \dots, R_{n_2+2})$ is taken as the initial solution for the backtrack. The parallel class R_i will be chosen in

$$\mathcal{C}_1 = \{R_z \in \mathcal{C} \mid M(1, z) = \mathcal{T}_2\} \quad \text{for} \quad i = n_1 + 3, \dots, n_1 + n_2 + 1$$

and

$$\mathcal{C}_2 = \{R_z \in \mathcal{C} \mid M(1,z) = \mathcal{T}_3\} \quad \text{for} \quad i = n_1+n_2+2,\ldots,n_1+n_2+n_3+1.$$

Clearly, $|\mathcal{C}_1| = \binom{6}{5}\binom{6}{1} = 36$ and $|\mathcal{C}_2| = \binom{6}{4}\binom{6}{2} = 225$. For each choice of these parallel classes, we check that each partial solution could lead simultaneously to a resolution of type s of a 2-$(12,6,5c)$ design and of a 3-$(12,6,2c)$ design. We perform isomorph rejection only at the end of the search and at the level given by the parallel class n_1+n_2+1. Also, at the end of the search, we check that the designs found are consistent with Lemma 2 with respect to the parallel class $R_1 = (X,Y)$. In both levels, we use the full automorphism group. A computation search shows the following result.

Theorem 3 *Let $\mathcal{R}$ be a resolution of type s for a* 2-$(12,6,15)$ *design. Then the number of nonisomorphic initial structures of type s for $\mathcal{R}$ is equal to zero for $s=2,5$, and there exist respectively* 1, 9, 21987 *and* 793 *initial structures for s = 1, 3, 4 and 6. While for a* 2-$(12,6,10)$ *design, the number of nonisomorphic initial structures of type s is equal to zero for $s=2$, and there are respectively* 1, 6 *and* 1581 *initial structures for $s=1$,* 3 *and* 4.

The CPU time to generate the initial structures of the above Theorem was approximately 2 and 14 days for $c=2$ and $c=3$, respectively. However for $c=3$, our computer program was not able to generate all such designs in a reasonable time, for $s=7,8,9$. For example, for $s=7$, the program does not finish even after 34 days. During this running time the program generated more than twenty million initial structures. Preliminary computational experiments show that for $s=8$ and $s=9$ the CPU time and the number of possible initial structures seem to grow even more. Additional computational experiments with some initial structures show that it is not feasible to enumerate all simple resolutions with the existing computational resources.

Sometimes, sets will be denoted without brackets and commas, thus, for example, the set $\{x,y,z\}$ will be denoted simply by xyz.

Let $\mathcal{F}$ be any initial structure of type s of a resolution of a 2-$(12,6,5c)$ design constructed by our backtracking. Denote by $\ell(W)$ the number of occurrences of a subset W of X in the design $\mathcal{F}$. We define

$$I(\mathcal{F}) = \mathcal{F} \cup (\underbrace{(012,345),\ldots,(012,345)}_{2c-\ell(012)},\ldots,\underbrace{(045,123),\ldots,(045,123)}_{2c-\ell(045)}). \quad (3)$$

Note that each 3-subset of X occurs in $I(\mathcal{F})$ $2c$ times. (Abusing the notation, each pair of triples of elements of X also will be called *parallel class*).

For the structure $I(\mathcal{F})$ we define a vector (p_1, p_2, p_3) where

$$\begin{aligned} p_1 = |\mathcal{F}| + 2c - \ell(012), \quad & p_2 = p_1 + 3c - \ell(01) + \ell(012), \\ & p_3 = p_2 + 3c - \ell(02) + \ell(012). \end{aligned} \tag{4}$$

This vector will be used in the following section. Note that the first p_1, p_2, and p_3 parallel classes of $I(\mathcal{F})$ contain all the occurrences of the sets 012, 01, and 02, required in a resolvable 2-$(12, 6, 5c)$ design, respectively.

For example, up to isomorphism, the only structure $I(\mathcal{F})$ of type s, where s is the minimal element of $S(c)$, for a 2-$(12, 6, 5c)$ design has the form:

$$\begin{array}{lll}
(c) & (\{0,1,2,3,4,5\}, & \{6,7,8,9,10,11\}) \\
(c) & (\{0,1,2\}, & \{3,4,5\}) \\
(c) & (\{0,1,3\}, & \{2,4,5\}) \\
(c) & (\{0,1,4\}, & \{2,3,5\}) \\
(c) & (\{0,1,5\}, & \{2,3,4\}) \\
(c) & (\{0,2,3\}, & \{1,4,5\}) \\
(c) & (\{0,2,4\}, & \{1,3,5\}) \\
(c) & (\{0,2,5\}, & \{1,3,4\}) \\
(c) & (\{0,3,4\}, & \{1,2,5\}) \\
(c) & (\{0,3,5\}, & \{1,2,4\}) \\
(c) & (\{0,4,5\}, & \{1,2,3\})
\end{array}$$

Here, the (c) in the first column means that the pair (A, B) appears c times. For this example, $p_1 = 2c$, $p_2 = 5c$, and $p_3 = 8c$.

4 Extending Initial Structures

This section outlines the backtrack algorithm over parallel classes with partial isomorph rejection used for determining all extensions of the initial structures for a 2-$(12, 6, 5c)$ design.

In the previous section we divided the classification of resolutions of a 2-$(12, 6, 5c)$ design into subproblems of type $1, 2, \ldots,$ and $|S(c)|$. We now divide the classification problem of resolutions of type s $(1 \leq s \leq |S(c)|)$ of a 2-$(12, 6, 5c)$ design into smaller problems. Let m_s denote the number of nonisomorphic initial structures of type s of a 2-$(12, 6, 5c)$ design. We label these initial structures as $\mathcal{F}_1, \ldots, \mathcal{F}_{m_s}$. Thus we say that a resolution $\mathcal{R}$ of a 2-$(12, 6, 5c)$ design of type s has *subtype* j $(1 \leq j \leq m_s)$, if $\mathcal{R}$ has at least a subcollection of parallel classes isomorphic to $\mathcal{F}_j$, but has no subcollection of parallel classes isomorphic to an initial structure $\mathcal{F}_\ell$ with $\ell < j$. It is easy to see that two resolutions of type s of different subtypes cannot be isomorphic. This proves that the classification of every resolution of type s

and subtype $1 \leq j \leq m_s$ implies the classification of all resolutions of type s.

Now we describe the backtrack algorithm over parallel classes used for determining all extensions of the initial structure $\mathcal{F}_j$ $(1 \leq j \leq m_s)$ of type s to resolutions of type s and subtype j of a 2-$(12, 6, 5c)$ design. This structure is used as the initial solution for the backtrack search. Assume that $(R_1, \ldots, R_{i-1})$ is a partial solution. Then the ith parallel class will be chosen in the set $\mathcal{P}(A_i, B_i)$, where (A_i, B_i) is the ith element of $I(\mathcal{F}_j)$ (see (3)). For each choice of these parallel classes, we check that each partial solution could lead simultaneously to a resolution of type s and subtype j of a 2-$(12, 6, 5c)$ design and of a 3-$(12, 6, 2c)$ design.

A partial isomorph rejection scheme [13] is employed to avoid processing isomorphic subproblems in the backtrack tree. Such isomorph rejection is most useful if it can be applied to intermediate levels of the search tree. We perform isomorph rejection only at the top and at selected intermediate levels. These levels are called *testing levels*. For the initial structure $\mathcal{F}_j$ of type s, the intermediate testing levels are given by the vector (p_1, p_2, p_3), see (4). A partial isomorph rejection at the intermediate level p_i consists in generating the set $\mathcal{A}(p_i)$ of all the partial solutions $(R_1, \ldots, R_{p_i})$, and choosing only one partial solution from each isomorphism class of $\mathcal{A}(p_i)$ to be extended in the search tree. This partial solution is called a *certificate*. For these intermediate levels we use the automorphism group that leaves invariant the set X. This partial isomorph rejection scheme reduces substantially the computer time required for generating resolutions a of 2-$(12, 6, 5c)$ design (see [10]). We also perform isomorph rejection at the end of the search, in this case using the full automorphism group.

In order to determine the certificates in every intermediate testing level p for the initial structure $\mathcal{F}_j$, we use the package Nauty due to McKay [12] as follows. When a partial solution $\mathcal{R}_p = (R_1, \ldots, R_p)$ is generated, we construct the bipartite point-block incidence graph $G(\mathcal{R}_p)$ and then call Nauty to get the canonical form of $G(\mathcal{R}_p)$. Hence the design $\mathcal{R}_p$ is a certificate in this testing level if the canonical form of $G(\mathcal{R}_p)$ was not generated before. Note that a certificate, in our context, is a collection $(R_1, \ldots, R_p)$ of parallel classes generated by our backtracking algorithm. The isomorph rejection described here is closely related to that in [10]. However, here the graph $G(\mathcal{R}_p)$ for calculating certificates has $2p$ vertices, whereas in [10] at every testing level it is used a bipartite point-block incidence graph with $2r$ vertices. This reduces the time needed to calculate certificates at intermediate testing levels.

Let $\mathcal{R} = (\mathcal{F}, R_{|\mathcal{F}|+1}, \ldots, R_p)$ and $\mathcal{Q} = (\mathcal{F}, Q_{|\mathcal{F}|+1}, \ldots, Q_p)$ be two partial solutions constructed by our backtrack search via an initial structure $\mathcal{F}$, where p is an intermediate testing level. Let $\mathcal{B} = \{(A_{p+1}, B_{p+1}), \ldots, (A_r, B_r)\}$ be the last $r - p$ parallel classes of $I(\mathcal{F})$ (see (3)). Now assume that there exists an isomorphism α mapping $G(\mathcal{R})$ to $G(\mathcal{Q})$ that leaves invariant the set X. ¿From [2, Prop. 9.42], α induces an isomorphism from $\mathcal{R}$ to $\mathcal{Q}$. Since $\alpha(X) = X$, it follows that $\alpha(\mathcal{F}) = \mathcal{F}$. Next we show that $\alpha(\mathcal{B}) = \mathcal{B}$. Let T be a 3-subset occurring in $\mathcal{B}$. By definition of the intermediate testing level p, T does not occur in $\mathcal{R} - \mathcal{F}$. This implies that T and $\alpha(T)$ occur the same number of times in $\mathcal{F}$. Hence, by (3), T and $\alpha(T)$ occur the same number of times in $\mathcal{B}$. This implies that $\alpha(\mathcal{B}) = \mathcal{B}$. Thus, the proof of the following lemma is similar to that of [10, Theorem 6].

Lemma 4 *Let $\mathcal{F}_j$ $(1 \leq j \leq m_s)$ be an initial structure of type s. Then, our isomorph rejection algorithm generates, without repetition, all nonisomorphic resolutions of type s and subtype j of a* 2-$(12, 6, 5c)$ *design.*

Since two resolutions of type s having different subtypes are nonisomorphic, Lemma 4 implies the following theorem.

Theorem 5 *For any type $s \in \{1, \ldots, |S(c)|\}$, our isomorph rejection algorithm generates, without repetition, all nonisomorphic resolutions of type s of a* 2-$(12, 6, 5c)$ *design.*

5 Pruning

As in any backtrack search, it is useful to identify when a partial solution cannot lead to a solution although it is still possible to extend the partial solution further. Detecting and pruning immediately such partial solutions can avoid a lot of work compared with the cost of detection. We now describe how we used Alltop's Theorem to obtain a new pruning technique for the enumeration of resolvable $(2k, k, \lambda)$ designs. It also uses a relation between 2-subsets and 3-subsets of points of the design.

Let $\mathcal{R} = (R_1, \ldots, R_r)$ be any resolution of a $(2k, k, \lambda)$ design. For $1 \leq j \leq r$, let $c_j(ab)$ and $t_j(xyz)$ be the number of occurrences of ab and xyz in $(R_1, \ldots, R_j)$, respectively. Let $f_j(ab) = \lambda - c_j(ab)$ and $f_j(xyz) = \lambda' - t_j(xyz)$, where $\lambda' = \lambda(k-2)/(2k-2)$ (see Theorem 1). Moreover, for each subset S of $\{x, y, z\}$, we define

$$n_j(S, ab) = |\{R \in (R_{j+1}, \ldots, R_r) : S \cup \{a, b\} \text{ occurs in the parallel class } R\}|.$$

We denote by m the number of parallel classes R in $(R_{j+1}, \ldots, R_r)$ such that the triple $\{x, y, z\}$ occurs in R but the points a and b occur in different blocks of R. By counting the occurrences of the sets $\{a, b\}$, $\{x, y, z\}$,

$\{a,b,x\}$, $\{a,b,y\}$ and $\{a,b,z\}$ in $(R_{j+1},\ldots,R_r)$, we get that

$$\begin{aligned} n_j(xyz,ab)+n_j(xy,ab)+n_j(xz,ab)+n_j(x,ab) &= f_j(abx) \\ n_j(xyz,ab)+n_j(xy,ab)+n_j(yz,ab)+n_j(y,ab) &= f_j(aby) \\ n_j(xyz,ab)+n_j(xz,ab)+n_j(yz,ab)+n_j(z,ab) &= f_j(abz) \\ n_j(xyz,ab)+n_j(\emptyset,ab)+m &= f_j(xyz) \\ \sum_{S\subset\{x,y,z\}} n_j(S,ab) &= f_j(ab) \end{aligned} \tag{5}$$

and

$$m \leq r-j-f_j(ab). \tag{6}$$

This Diophantine linear system is used to prune the search tree as follows. Clearly in each partial solution $(R_1,\ldots,R_j)$ with $j\leq r$ we can calculate $f_j(abx)$, $f_j(aby)$, $f_j(abz)$, $f_j(xyz)$ and $f_j(ab)$ for any five points a, b, x, y and z. However, the eight numbers $n_j(S,ab)$ $(S\subset\{x,y,z\})$, and m are unknown. Thus for any partial solution, we have a Diophantine linear system with five equations and an inequality. Then any partial solution $(R_1,\ldots,R_j)$ could lead to a resolution for a $(2k,k,\lambda)$ design if for any five different points a, b, x, y and z, there exists a nonnegative integer solution

$$(n_j(xyz,ab),n_j(xy,ab),n_j(xz,ab),n_j(x,ab),\ldots,n_j(\emptyset,ab),m)$$

for the system (5)-(6).

Example. For $k=6$ and $\lambda=15$, consider the following partial solution $(R_1,\ldots,R_4)$:

$$\begin{array}{ll} (\{0,1,2,3,4,5\}, & \{6,7,8,9,10,11\}) \\ (\{0,1,2,3,4,6\}, & \{5,7,8,9,10,11\}) \\ (\{0,1,2,4,5,9\}, & \{3,6,7,8,10,11\}) \\ (\{0,1,2,4,5,10\}, & \{3,6,7,8,9,11\}). \end{array}$$

In this partial solution, for the five points $a=4$, $b=5$, $x=0$, $y=1$ and $z=2$, we have

$$f_4(4,5)=12,\ f_4(0,1,2)=2,\ f_4(4,5,0)=3,\ f_4(4,5,1)=3,\ f_4(4,5,2)=3.$$

However, an exhaustive search shows that there is no nonnegative integer solution for the linear system (5). Hence this partial solution cannot lead to a resolution for a 2-$(12,6,15)$ design. Thus this node (partial solution) is pruned in the search tree.

More constraints at each node will result in the search tree containing fewer nodes, but the overall cost may be higher. To avoid a high cost in the search, we just check out the constraints given by $0\leq f_j(abx)\leq f_j(aby)\leq f_j(abz)\leq 2$ and $0\leq f_j(xyz)\leq 2$. Moreover, since the solutions of system (5) do not depend on j, they are calculated once at the beginning of the computation search.

6 Results

The backtracking algorithms described in this work were implemented in the *C* language. Tables 2 and 3 present respectively the number of all nonsimple and a subfamily of simple resolutions for 2-$(12, 6, 15)$ designs and the sizes of their automorphism groups. Table 4 presents the number of all resolutions for 2-$(12, 6, 10)$ designs and the sizes of their automorphism groups.

For resolvable 2-$(12, 6, 15)$ designs, the program for extending the initial structures ran for approximately 2.1 years on a network of 8 PCs with a total of 22 Opteron processors running at 2 GHz; while for resolvable 2-$(12, 6, 10)$ designs, the execution time was about 3 minutes.

Since any resolution of a 2-$(12, 6, 15)$ design is nonsimple if and only if it is of type either 1, 3, or 4, Table 2(a)-(c) proves the next Theorem.

Theorem 6 *There are* $58, 619, 820, 853$ *nonisomorphic nonsimple resolvable* 2-$(12, 6, 15)$ *designs.*

Table 3 shows the following result.

Theorem 7 *There are at least* $30, 885, 758, 702$ *nonisomorphic simple resolvable* 2-$(12, 6, 15)$ *designs.*

Table 4 shows the next result.

Theorem 8 *There are* 545 *nonisomorphic resolvable* 2-$(12, 6, 10)$ *designs.*

Also our algorithm confirms that there exists one nonisomorphic resolvable 2-$(12, 6, 5)$ design. The size of its automorphism group is 7920.

Table 2: Classification of nonsimple resolvable 2-$(12, 6, 15)$ designs

(a) Resolutions of type 1

$\|Aut(D)\|$	1	2	3	4	5	6	8	9	10	12
Nr	21112878	39456	684	822	29	147	59	6	6	25

16	18	20	24	36	48	60	72	120	216	432	720	7920	Total
8	6	2	11	3	1	1	1	1	1	1	1	1	21154150

Table 2 (continued)

(b) Resolutions of type 3

$\lvert Aut(D)\rvert$	1	2	3	4	5	6	8	10	12	20	55	Total
Nr	2398771058	326090	84	1477	88	10	21	9	1	4	1	2399098843

(c) Resolutions of type 4

$\lvert Aut(D)\rvert$	1	2	3	4	6	8	9	12	16	18	24	27	36	48	54	144	432	Total
Nr	56197075420	2475135	5945	10609	237	415	3	43	25	6	13	1	1	3	2	1	1	56199567860

Table 3: The resolvable simple 2-(12, 6, 15) designs of type 6

$\lvert Aut(D)\rvert$	1	2	3	4	6	8	12	Total
Nr	30885043846	712554	1040	1195	41	23	3	30885758702

7 Consistency checking

Validation of the results is very important and essential in any computer-aided classification, see [6] for a survey of validations methods. Among these methods, consistency checking ([8]) has been used successfully in some recent classification studies [5], [7].

We perform a consistency check based on double counting. On the one hand, we rely on the classified resolvable 2-(12, 6, 10) designs. On the other hand, we rely on data obtained in the extension stage. Since our classification of resolvable 2-(12, 6, 5c) designs is based on the classification of resolutions of type s ($1 \leq s \leq |S(c)|$), it is sufficient to perform a consistency checking on each classification of resolutions of type s, for $s = 1, \ldots, |S(c)|$.

Let $N_{s,g}$ be the number of nonisomorphic resolutions of type s whose automorphism group has order g. Then, by the orbit-stabilizer theorem,

Table 4: Classification of nonsimple resolvable 2-$(12, 6, 10)$ designs

(a) Resolutions of type 1

$\lvert \mathrm{Aut}(D) \rvert$	1	2	3	4	6	8	10	12	16	18	24	32	36	48	240	1440	7920	Total
Nr	15	36	1	21	7	5	4	5	9	2	2	4	2	2	1	2	1	119

(b) Resolutions of type 3

$\lvert \mathrm{Aut}(D) \rvert$	1	2	3	4	6	8	10	16	40	110	240	Total
Nr	198	141	3	25	2	12	3	5	1	1	1	392

(c) Resolutions of type 4

$\lvert \mathrm{Aut}(D) \rvert$	1	2	4	6	8	11	12	24	Total
Nr	9	15	3	2	1	1	1	2	34

the total number of resolutions of type s of a 2-$(12, 6, 5c)$ design is

$$\sum_{g \geq 1} \frac{12!\, N_{s,g}}{g}.$$

For the extension stage the counting is done as follows. Let the initial structures be $\mathcal{F}_j$, $1 \leq j \leq m_s$, of type s of a 2-$(12, 6, 5c)$ design. Let $E(\mathcal{F}_j, h)$ be the set of all resolutions $\mathcal{R}$ that would be generated by the backtrack algorithm without isomorph rejection as extensions of $\mathcal{F}_j$ such that $\mathcal{R}$ has h subcollections of parallel classes isomorphic to $\mathcal{F}_j$. Since $\mathcal{F}_j$ itself belongs to $\mathcal{R}$, it follows that $h \geq 1$. Let $M_{j,h} = |E(\mathcal{F}_j, h)|$. Next, we will explain how to calculate this number during the backtrack search with isomorph rejection. Since any two resolutions of type s of different subtypes are not isomorphic, it follows from the orbit-stabilizer theorem that the total number of resolutions of type s of a 2-$(12, 6, 5c)$ design is

$$\sum_{h \geq 1} \frac{1}{h} \sum_{j=1}^{m_s} \frac{12!\, M_{j,h}}{|Aut(\mathcal{F}_j)|},$$

where the division by h is required because the inner sum counts every resolution once for each of the h subcollections of parallel classes isomorphic to $\mathcal{F}_j$ that occur in a resolution.

Both counts give the same results for each classification of resolutions of type s, for $s = 1, \dots, |S(c)|$, for $c = 2$ and $c = 3$. This gives us confidence that the classifications are correct.

Now, we explain how to calculate the number $M_{j,h}$ during the backtrack search with isomorph rejection. At each testing level p, together with a certificate $\mathcal{R}_p$, we store the number, $M(\mathcal{R}_p)$, of all partial solutions that would be generated by the backtrack algorithm without isomorph rejection that are isomorphic to $\mathcal{R}_p$. Clearly, for the testing level p_1, $M(\mathcal{R}_{p_1})$ is equal to the size of the isomorphism class of $\mathcal{R}_{p_1}$. Suppose that $i > 1$. For each partial solution $\mathcal{Q}_{p_i}$ generated from a certificate $\mathcal{Q}_{p_{i-1}}$, let $N(\mathcal{Q}_{p_i}) = M(\mathcal{Q}_{p_{i-1}})$. Then, it is not hard to see that, for every certificate $\mathcal{R}_{p_i}$

$$M(\mathcal{R}_{p_i}) = \sum_{\mathcal{Q}_{p_i} \text{is isomorphic to } \mathcal{R}_{p_i}} N(\mathcal{Q}_{p_i}).$$

In consequence, if $C(h)$ is equal to the set of all certificates at the top level having h subcollections of parallel classes isomorphic to $\mathcal{F}_j$, then

$$M_{j,h} = \sum_{\mathcal{R} \in C(h)} M(\mathcal{R}).$$

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