Orthogonal polygon illumination with rotating floodlights

J. Aguilar[1], I. Aldana[1], C. Cedillo[2], D. Flores-Peñaloza*[3], E. Solís[1], J. Urrutia†[1], and C. Velarde[4]

[1] Instituto de Matemáticas, UNAM, México.
[2]Universidad Autónoma Metropolitana, Unidad Iztapala, México
[3]Departamento de Matemáticas, Facultad de Ciencias, UNAM, México.
[4]Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, UNAM, México.

Abstract

We consider a variation on the classic art gallery problem on orthogonal and simple polygons. In our setting, our guards have a bounded angle of visibility, and they rotate uniformly. For orthogonal polygons, we study the case of illuminating the interior or the exterior of a polygon with or without holes with rotating floodlights of size $\frac{3\pi}{2}$. We also study the problem of illuminating simple polygons using rotating floodlights of size π.

1 Introduction

In 1973, V. Klee [5] posed the problem of determining how many *lights* are always sufficient to *illuminate* the interior of an art gallery represented by a simple polygon on the plane. V. Chvátal proved that $\lfloor \frac{n}{3} \rfloor$ lights are always sufficient an sometimes necessary. Shortly thereafter Fisk [8] gave an elegant and simple proof of Chvátal's result. In 1987, J. O'Rourke published the book *Art Gallery Theorems and Algorithms* [10], solely dedicated to the study of illumination problems of polygons in the plane. Since then, several variants on this original problem have been posed. Some of these variants involve illuminating some restricted families of simple polygons, e.g. orthogonal polygons, or using light sources with a restricted angle of illumination, called floodlights, or more recently modem illumination problems in which the signal emmited by a source (e.g. a wireless modem) can penetrate several edges of a polygon [7, 2]. The surveys of T. Shermer [11] and J. Urrutia [13] are good sources of information of most of the variations studied of the original Art Gallery problem.

An α-floodlight f is a light source that emits light within a cone of angular size α bounded by two rays emanating from a point, the apex of f, α is called the size of f. In 1997, Urrutia [4] proposed the problem of illuminating the plane with n floodlights of sizes $\alpha_1, \ldots, \alpha_n$ such that their apices have to be located at a given point set P with n elements, one apex per point. In [4] it was proved that this can always be done if $\alpha_1 + \cdots + \alpha_n = 2\pi$, and $\alpha_1, \ldots, \alpha_n \leq \pi$. In 1994, Estivill-Castro *et al.* [6] studied the following problem: How many $\pi/2$ vertex floodlights are always sufficient and sometimes necessary to illuminate an orthogonal polygon with n vertices? They proved that the answer to that question is precisely $\frac{3n-4}{8}$. In all of these variants the floodlights are static. In 2011, Urrutia posed the problem of illuminating the plane with rotating floodlights of a given size such that their apices have to be located at the elements of a point set P, see E. Kranakis *et al.* [9]. They studied the problem of determining the initial orientation and sizes of a set of rotating floodlights that always illuminate the plane. They proved, among other results, that three floodlights of size π are sufficient and necessary to illuminate the plane all the time, as the floodlights rotate counterclockwise at the same speed. In 2013, Bereg *et al.* [3] studied a similar problem in which the apices of rotating floodlights have to be located at the elements of a point set P, and the elements of a second point set must be always illuminated by at least one floodlight.

We will assume that all the lights rotate counterclockwise such that at a time $t = \alpha$, they have rotated α degrees. We say that a point p of the plane is illuminated by a floodlight f at instant t (w.r.t some polygon P), if the segment $\overline{fp}$ is contained in the illumination angle of f at time t, and there is no edge of P intersecting the interior of $\overline{fp}$. In this paper we study the following problems: i) illumination of orthogonal polygons with vertex floodlights of size $\frac{3\pi}{2}$ (i.e. located at the vertices of the a polygon). And ii) illumination of simple polygons with vertex floodlights of size π. For simplicity, we assume that all floodlights rotate counter-clockwise. We give sharp bounds for both problems.

*Partially supported by grants 168277 (CONACYT, México), and IA102513 (PAPIIT, UNAM, México).
†Partially supported by grant SEP-CONACyT of México no. 80268.

2 Orthogonal polygons

We use the following notation as in [12]. Given an orthogonal polygon P, we call an edge of P a *top edge* if there is an $\epsilon > 0$ such that any point at distance less than or equal to ϵ from the mid-point of e, and below e belongs to the interior of P. *Right*, *bottom*, and *left* edges are defined in a similar way. A vertex of P is called a *top-left* vertex if the edges of P incident to it are a *top* and a *left* edge. *Top-right*, *bottom-right*, and *bottom-left* vertices are defined in a similar way; see Figure 1. Since a vertex may also be convex or reflex, there are eight possible types of vertices in an orthogonal polygon.

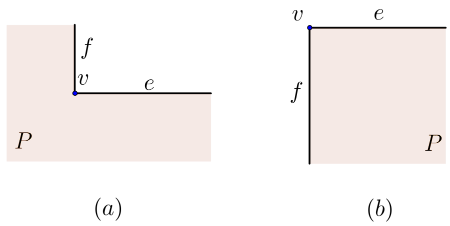

Figure 1: In (a) e is a *top* edge, f is a *right* edge, and v is a *TR*-vertex. In (b) f is a *left* edge, and v is a *TL*-vertex.

For a given rotating floodlight f, the *beginning of* f is the oriented half-line starting at the apex of f, that leaves the area illuminated by f to its right, and the area not illuminated by f to its left. The *end* of f is defined in a similar way, see Figure 2.

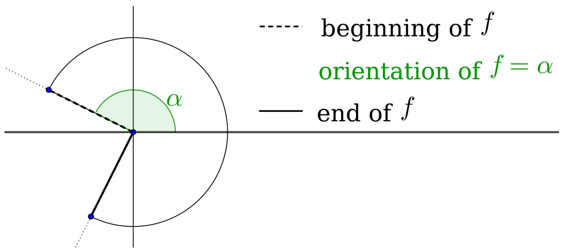

Figure 2: Beginning, end and orientation of a floodlight.

Given a rotating floodlight f, its *orientation* at time t is the value of the (non-negative) angle made from the positive x-axis to the *beginning* of f, for illustration see Figure 2.

Following [1] we define placement rules for α-floodlights at the vertices of an orthogonal polygon as follows: in the *Top-Left*-illumination rule (for short the TL-illumination rule), we place an α-floodlight at the left endpoint of each top edge, and an α-floodlight at the top vertex of each left edge. Unless otherwise specified, we will assume that the orientation of our α-floodlights is 0. For example if $\alpha = \pi$, in the initial position of a π-floodlight, it will illuminate the half-plane below the line passing through its apex.

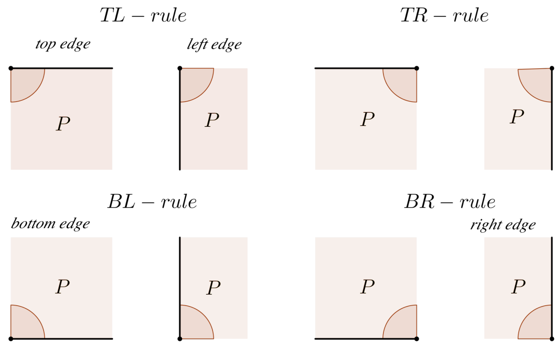

Figure 3: *TL*-rule for $\frac{\pi}{2}$-floodlights.

The other three possible rules (*TR*-rule, *BR*-rule, and *BL*-rule) are defined in a similar way, see Figure 3, although the orientations of the floodlights are, respectively $\frac{3\pi}{2}$, $\frac{2\pi}{2}$, and $\frac{\pi}{2}$.

Now we define the α-(TR,BL)-illumination rule as the union of the α-floodlights of the TR-illumination rule, and the BL-illumination rule, with the additional restriction that all the α-floodlights start with equal orientations. The α-(TL,BR)-illumination rule is defined similarly. The following result proved in [1] will be used.

Theorem 1 *[1] Let P be an orthogonal polygon with or without holes. Then the TL-illumination rule produces an assignment of static floodlights of size $\frac{\pi}{2}$, that illuminate the interior of P.*

The same is true for each of the *TR*-illumination rule, the *BR*-illuminatio rule and the *BL*-illumination rule.

3 Guarding the interior of orthogonal polygons

The main result of this section is the following Theorem.

Theorem 2 *Let P be an orthogonal polygon with $2m$ vertices, with or without holes. Then, m $\frac{3\pi}{2}$-rotating floodlights, with initial orientation 0, and located according to the $\frac{3\pi}{2}$-(TR,BL)-illumination rule or the $\frac{3\pi}{2}$-(TL,BR)-illumination rule, are always sufficient to illuminate P. Furthermore m $\frac{3\pi}{2}$-rotating floodlights are sometimes necessary.*

Proof. Let f be a $\frac{3\pi}{2}$-rotating floodlight, placed at a vertex v of P by the TL-illumination rule. We say that f is active at a time t according to the TL-illumination rule if f illuminates the region that would be illuminated by a (non-rotating) $\frac{\pi}{2}$-floodlight placed at p by the TL-illumination rule. In a similar way we define an active $\frac{3\pi}{2}$-rotating floodlight according to the TR-, BR-, and BL-illumination rules.

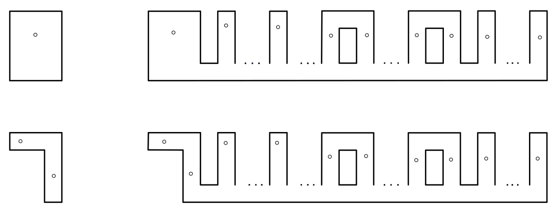

Figure 4: Orthogonal polygons that need m $\frac{3\pi}{2}$-rotating floodlight to be illuminated.

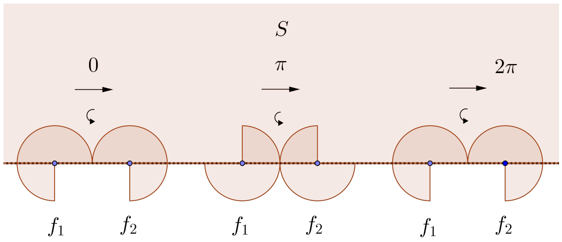

Figure 5: Floodlights f_1 and f_2 work together to illuminate the semi-plane S. It is shown how both of them illuminate for a period of time 2π.

If a rotating $\frac{3\pi}{2}$-floodlight f was placed according to the TL-illumination rule, then f is active according to the TL-illumination rule in the interval $[0, \pi]$. In a similar way we can prove that if f was placed according to the BR-illumination rule, it will be active in the interval $[\pi, 2\pi]$. Thus rotating floodlights placed according to the the $\frac{3\pi}{2}$-*(TL,BR)*-illumination rule will always illuminate P. The second case is proved in a similar way.

To prove that the above bound is tight, we can see that the orthogonal polygons with $2m$ vertices shown in Figure 4, requires m rotating $\frac{3\pi}{2}$-rotating floodlight to be illuminated.

□

4 Guarding the exterior of orthogonal polygons

In this section we give tight bounds on the number of floodlights needed to illuminate the exterior of a polygon. The following lemma is given without proof:

Lemma 3 *Let S be a semiplane. Two $\frac{3\pi}{2}$-rotating floodlights placed on a horizontal line ℓ are sufficient and necessary to illuminate the half-plane above ℓ. See Figure 5.*

Theorem 4 *Let P be an orthogonal polygon with $2m$ vertices, with or without holes. Then, $m+2$ $\frac{3\pi}{2}$-rotating floodlights, located at the vertices of P, are always sufficient, and sometimes necessary, to illuminate the exterior of P.*

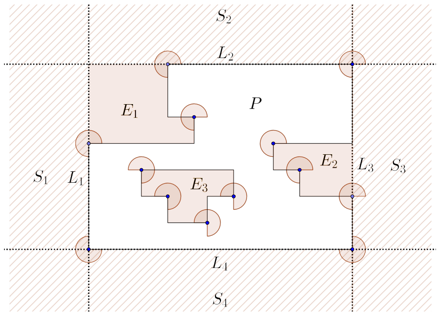

Figure 6: Illuminating the exterior of an orthogonal polygon.

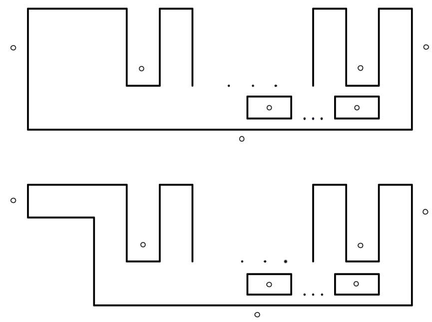

Figure 7: Polygons requiring $m+2$ $\frac{3\pi}{2}$-rotating floodlights to illuminate their exterior.

Proof. (sketch) Let B be the smallest rectangle that contains P, and let L_1, L_2, L_3 and L_4 be its supporting lines, see Figure 6. We illuminate first the exterior of B using Lemma 3. Observe that $B \setminus P$ is a set of orthogonal polygons, each of which is illuminated according to Theorem 2. It is easy to see that, in total we use $m+2$ floodlights. The details are omitted for lack of space.

To prove that the above bound is tight, we can see that the orthogonal polygons with $2m$ vertices illustrated in Figure 7, requires $m+2$ floodlights to be illuminated. □

5 Guarding the interior of simple polygons with rotating floodlights of size π

In this section we study the problem of illuminating simple polygons with rotating floodlights of size π, located on vertices of the polygon. We give tight bounds on the number of π-rotating floodlights needed to illuminate the interior of a simple polygon.

The following Lemma is given without a proof.

Lemma 5 *Three π-rotating floodlights are sufficient and necessary to illuminate a triangle T.*

It is not hard to verify that in order to illuminate T, the three π-rotating floodlights, must start with the

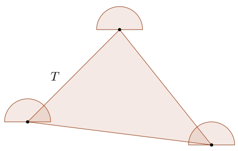

Figure 8:

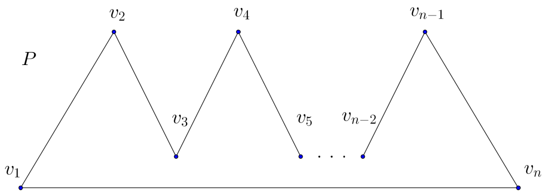

Figure 9: Polygons whose number of required floodlights reaches the lower bound. Note that each peak of a polygon requires three floodlights to be illuminated.

same orientations, for otherwise at some time t, they will fail to illuminate T, see Figure 8. We proceed to show now the main result in this section.

Theorem 6 *Let P be a simple polygon with n vertices. Then, n rotating π-floodlights, located at the vertices of P, are always sufficient, and sometimes necessary, to illuminate P.*

Proof. Place at each vertex of P a rotating π-floodlight such that all of them start with the same orientation, and consider a triangulation $T(P)$ of P. By Lemma 5, each triangle in T will be illuminated. Therefore P is illuminated.

Consider now the family of polygons with $2m+1$ vertices shown in Figure 9. It is easy to see that the interior of the triangle with vertices $v_{2i+1}, v_{2i+2}, v_{2i+3}$ can be illuminated only with the floodlights at v_{2i+1}, v_{2i+2}, and v_{2i+3}, $i = 1, \ldots, m-1$. It is easy to verify that in order to illuminate the triangles with vertices v_1, v_2, and v_3, and the triangle with vertices v_{2m-1}, v_{2m}, and v_{2m+1} we also need a rotating π-floodlight at each of these vertices. $\square$

6 Conclusions and future work

We have given sharp bounds on the number of rotating vertex α-floodlights required to illuminate orthogonal and simple polygons. All of our results lead trivially to linear time algorithms to find the vertices of the polygons where the floodlights have to be located, as well as their initial orientations.

In a forthcoming paper we will prove that the problem of finding the smallest number of rotating α-floodlights required to always illuminate a given polygon is NP-complete.

References

[1] J. Abello, V. Estivill-Castro, T. Shermer, and J. Urrutia. Illumination of orthogonal polygons with orthogonal floodlights. *Int. J. Comput. Geom. Appl.*, 8(1):25–38, 1998.

[2] O. Aichholzer, R. Fabila-Monroy, D. Flores-Peñaloza, T. Hackl, C. Huemer, J. Urrutia, and B. Vogtenhuber. Modem illumination of monotone polygons. *In Proc. 25th European Workshop on Computational Geometry EuroCG '09*, pages 167–170, 2009.

[3] S. Bereg, J. Miguel Díaz-Báñez, M. Fort, M. Lopez, P. Pérez-Lantero, and J. Urrutia. Continuous surveillance of points by rotating floodlights. *Int. J. Comput. Geom. Appl.*, to appear.

[4] P. Bose, L. Guibas, A. Lubiw, M. Overmars, D. Souvaine, and J. Urrutia. The floodlight illumination problem. *Int. Journal in Computational Geometry*, 7:153–163, 1997.

[5] V. Chvátal. A combinatorial theorem in plane geometry. *J. Combinatorial Theory Series B*, 18:39–41, 1975.

[6] V. Estivill-Castro and J. Urrutia. Optimal floodlight illumination of orthogonal art galleries. *Proc. of the Sixth Canadian Conference in Computational Geometry*, pages 81–86, 1994.

[7] R. Fabila-Monroy, A. Ruiz-Vargas, and J. Urrutia. On modem illumination problems. *XIII Encuentros de Geometría Computacional, Zaragoza, España*, pages 9–19, jun 2009.

[8] S. Fisk. A short proof of chvátal's watchman theorem. *Journal of Combinatorial Theory Series B*, 24(3):374, 1978.

[9] E. Kranakis, F. MacQuarie, O. Morales, and J. Urrutia. Uninterrupted coverage of a planar region with rotating directional antennae. *11th Int. Conf. on Ad-Hoc Networks and Wireless*, 2011.

[10] J. O'Rourke. *Art Gallery Theorems and Algorithms.* Oxford University Press, New York, 1987.

[11] T. Shermer. Recent results in art galleries. *Proc. IEEE*, 80(9):1384 – 1399, 1992.

[12] J. Urrutia. *Sixth proof of the Orthogonal Art Gallery Theorem.* Technical report TR-97-03, Department of Computer Science, University of Ottawa, 1997.

[13] J. Urrutia. Art gallery and illumination problems. In *Handbook of Computational Geometry*, pages 973–1027, North-Holland, 2000. Sack and J. Urrutia, eds.