Strong Chromatic Illumination of Orthogonal Polygons and Polyhedra with $\pi/2$- and π-floodlights and segments

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1 Introduction

Let P be an orthogonal polygon (polyhedron) in $\mathbb{R}^2$ ($\mathbb{R}^3$). We say that two points $p, q \in P$ are orthogonally visible if the smallest axis-aligned box (an axis-aligned rectangle in $\mathbb{R}^2$ or an axis-aligned cuboid in $\mathbb{R}^3$) containing them is contained in P. We consider a chromatic variation of the Art Gallery Problem on orthogonal polygons and orthogonal polyhedra under orthogonal visibility. A point p is illuminated by a point q if it is orthogonally visible from q. A set of points G illuminates P if every point in P is orthogonally visible from at least one element of G. In this paper we will assume that the elements of G have been assigned a color. From now on we will refer to orthogonal visibility simply as visibility.

A set G of colored points of a polygon or polyhedron P *strongly illuminates* P if every element p of P is visible from at least one element of G, and all the elements of G that see p have different color. We want to find the smallest number $\chi(n)$ of colors such that any n-vertex polygon or polyhedron can be strongly illuminated with a set of points using $\chi(n)$ colors. In this paper we will be using α-floodlights, or their generalizations in $\mathbb{R}^3$ to illuminate our polygons or polyhedron.

In the plane an α*-floodlight* f is a light source that emits light within a cone of angular size α bounded by two rays emanating from a point p, called the apex of f. In this paper, we will be dealing with α-floodlights of sizes π and $\pi/2$. In most of the cases we show how to illuminate the interior, the exterior, or the interior and the exterior of a polygon or polyhedron with α-floodlights or their generalization in $\mathbb{R}^3$.

2 Related work

In 1973, V. Klee posed the following problem: How many *lights* are always sufficient to *illuminate* the interior of an art gallery represented by a simple polygon on the plane with n vertices? V. Chvátal proved in [3] that $\lfloor \frac{n}{3} \rfloor$ lights are always sufficient and sometimes necessary. Since then, illumination problems have been studied by many authors. The book by J. O'Rourke [7], and the surveys by T. Shermer [8] and J. Urrutia [9] are good sources of information on art gallery problems.

Floodlight illumination problems were initially studied in 1997, see [2, 9]. A chromatic version of the problem was studied in [4]. The problem was motivated by applications in distributed robotics, where colors indicate the wireless frequencies assigned to a set of covering landmarks, so that a mobile robot can always communicate with at least one landmark without interference. A chromatic version using floodlights was studied in [6]. A chromatic version with conflict free illumination was studied in [1]. A chromatic version with conflict free illumination using guards with orthogonal visibility was studied in [5]. We present some of the results of the chromatic variant of the Art Gallery Problem in Table 1.

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Table 1: Previous Results

Bounds on the chromatic number				
Simple Polygons				
Polygon	lower	upper	C/V/α	Ref
Spiral		≤ 2	st/l/2π	[4]
Monotone	$\Omega(\sqrt{n})$		st/l/2π	[4]
General	$\Omega(n)$	$O(n)$	st/l/2π	[4]
Monotone		$O(\log n)$	cf/l/2π	[1]
General		$O(\log^2 n)$	cf/l/2π	[1]
General	1	1	st/l/$\leq \pi$	[6]
Orthogonal Polygons				
Stair		≤ 3	st/l/2π	[4]
Monotone	$\Omega(\sqrt{n})$		st/l/2π	[4]
General	$\Omega(\frac{\log^2 n}{\log^3 n})$		cf/l/2π	[5]
General	$\Omega(\log n)$	$O(\log n)$	st/l/2π	[1][5]
General	$\Omega(\log^2 n)$	$O(\log^2 n)$	cf/r/2π	[5]
C:Color type (cf:Conflict free st: Strong). V:Visibility model (l:standar r:orthogonal). α:Size of visibility.				

3 Preliminaries

We study first a chromatic variation of the Art Gallery Problem on *simple orthogonal polygons*. Observe that the internal angle at any vertex of an orthogonal polygon is of size $\pi/2$ or $3\pi/2$. A vertex with internal angle size $\pi/2$ is called a *convex vertex* and a vertex with internal angle size $3\pi/2$ is called a *reflex vertex*.

A *polyhedron* in $\mathbb{R}^3$ is a compact set bounded by a piecewise linear 2-manifold. A *face* of a polyhedron is a maximal planar subset of its boundary whose interior is connected and non-empty. A polyhedron is *orthogonal* if all of its faces are parallel to the xy-, xz- or yz-planes. The faces of an orthogonal polyhedron are orthogonal polygons with or without orthogonal holes. A *vertex* of a polyhedron is a vertex of any of its faces. An *edge* is a minimal positive-length straight line segment shared by two faces and joining two vertices of the polyhedron. A polyhedron P is a *lifting polyhedron* if there exists an *xy-plane* Z such that for all planes parallel to Z their intersection with P is either empty, or it is a vertical translation of $P \cap Z$.

For any polygon (polyhedron) P, $|P|$ denotes the number of vertices of P, ∂P, $\text{int}(P) = P - \partial P$, and $\text{ext}(P) = \mathbb{R}^2 - P$ ($\text{ext}(P) = \mathbb{R}^3 - P$) denote, respectively, the boundary, the interior and the exterior of P. $\chi(P, \alpha)$, $\chi(\text{ext}(P), \alpha)$, and $\chi(P \cup \text{ext}(P), \alpha)$ denote the smallest integer such that there is a set of α-guards, colored with $\chi(P, \alpha)$, $\chi(\text{ext}(P), \alpha)$, and $\chi(P \cup \text{ext}(P), \alpha)$ colors that strongly illuminates P, $\text{ext}(P)$, and $P \cup \text{ext}(P)$. For any point p the *visibility polygon* (*visibility polyhedron*) is the set of points visible from p.

Let P_1 and P_2 be two subpolygons (subpolyhedra) of P. We call P_1 and P_2 *independent* if no point in P can simultaneously see points from $\text{int}(P_1)$ and $\text{int}(P_2)$.

For a polygon P in the plane an edge e of P is a *right edge* if there is an $\varepsilon > 0$ such that any point at distance less than or equal to ε from any interior point of e and to the left of e belongs to the interior of P. *Left*, *top* and *bottom* edges are defined similarly. The *windows* of a subpolygon P' in P are those parts of $\partial P'$ that do not belong to ∂P. A window of P' is a *bottom window* in P if the window belongs to a bottom edge of P'. Similarly we define an *upper window*, a *left window* and a *right window*.

For a given floodlight f, the *beginning of* f is the oriented half-line starting at the apex of f, that leaves the area illuminated by f to its right, and the area not illuminated by f to its left. The *end* of f is defined in a similar way. Given a floodlight f, its *orientation* is the value of the (non-negative) angle between the positive x-axis to the *beginning* of f.

We proceed now to extend the concept of *floodlights* to $\mathbb{R}^3$. A *wedge* in $\mathbb{R}^3$ is the intersection, or the union of two halfspaces whose supporting planes intersect. The line of intersection of the supporting planes is called the *axis* of the wedge. A wedge is called *small*, if it is the intersection of two halfspaces. It is called *large* if it is the union of two halfspaces. Note that if a wedge $\mathcal{W}$ is small, then the intersection of $\mathcal{W}$ with a plane orthogonal to the axis of $\mathcal{W}$, determines an angular region $\mathcal{A}$ of size α less than or equal to π, if $\mathcal{W}$ is a big wedge, then α is greater than π. The wedge $\mathcal{W}$ will be called an α-wedge. An *orthogonal wedge* in $\mathbb{R}^3$ is the intersection or the union of two halfspaces whose supporting planes are orthogonal. If an orthogonal wedge is small, it is a $\frac{\pi}{2}$-wedge, if it is large it is a $\frac{3\pi}{2}$-wedge. An *α-segment* guard f of P placed on a segment s in P, guards all of the points of P visible from s and contained in an α-wedge whose axis contains s. We assume that an α-segment guard f can be rotated about its axis until it reaches a desired final orientation. In the rest of this paper we will assume that our α-segment guards are always placed in such a way that their supporting planes are parallel to the xy-, xz- or yz-planes of $\mathbb{R}^3$. We will use α-segment guards f such that they illuminate only points p within an α wedge, with the additional restriction that the shortest line segment joining p to f is a line segment orthogonal to f.

4 Orthogonally illuminating orthogonal polygons with floodlights of size $\pi/2$ and π

Theorem 1 *Let P be an orthogonal polygon with $|P| = n$. Then $\chi(P, \frac{\pi}{2}) = 1$.*

Proof. To prove our result, we will show how to illuminate P with a set of $\frac{\pi}{2}$-floodlights in such a way that no point in P is illuminated by two $\frac{\pi}{2}$-floodlights. Place $\frac{\pi}{2}$-floodlights on P using the following algorithm:

1. Place a $\frac{\pi}{2}$-floodlight f on the right vertex of a top edge of P with $3\pi/2$ orientation, and let P' be the area illuminated by this floodlight. Observe that since we are considering orthogonal visibility, P' is an orthogonal polygon.
2. Suppose $P' \neq P$, otherwise we are done. Then recursively place a $\frac{\pi}{2}$-floodlight on the right ver-

tex of every bottom window of P' with $3\pi/2$ orientation, increasing the illuminated area P'.

3. Continue this process recursively until P' has no more bottom windows. If $P' = P$ we are done.
4. Suppose that $P' \neq P$. Recursively proceed as follows: Each orthogonal subpolygon P'' of $P - P'$ has one or two edges containing windows of P'. In the first case, we proceed as follows: Suppose that P'' has a left edge e containing a right window of P'. Rotate P'' until e becomes a top edge, and repeat the process above starting at the right vertex of e. Proceed in a similar way with the top and the left windows of P'. In the second case, these two edges are incident to a vertex v of P''. Rotate P'' until v becomes part of a top edge, and restart the process at v from step one.

Observe that every floodlight placed in steps 1 and 3 is placed with $3\pi/2$ orientation on a bottom window, illuminating an area that is below P', not illuminated by f. Therefore no point in P' is illuminated by two floodlights. By the same reason, it is easy to see that no point in P is illuminated by two floodlights placed during the execution of Steps 2 and 3.

Using the same arguments we can see that in Step 4, when we illuminate the connected components of $P-P'$ no point in P is illuminated by two floodlights. Clearly at the end of our procedure the whole of P is illuminated. □

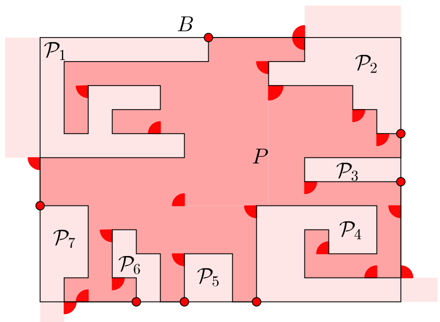


Figure 1: Illumination of the interior and exterior of a polygon with $\frac{\pi}{2}$-floodlights.

Theorem 2 *Let P be an orthogonal polygon with $|P| = n$. Then $\chi(ext(P), \frac{\pi}{2}) = 1$.*

Proof. Let B be the smallest bounding box of P. Let $\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_k\}$ be the set of polygons that are the connected components of $B - P$. To illuminate the exterior of P, we need to illuminate the polygons in $\mathcal{P}$ as well as the exterior of B. Consider first the polygons $\mathcal{P}_i \in \mathcal{P}$ such that one of their top edges belongs to the boundary of B, e.g. $\mathcal{P}_1$ in Figure 1. Illuminate these polygons using the algorithm in Theorem 1, and starting by placing a floodlight on its right endpoint.

In a similar way we can illuminate the orthogonal polygons in $\mathcal{P}$ containing a left, bottom, or right edge in B. Observe that while illuminating the polygons in $\mathcal{P}$, some of the light used to illuminate them will "spill out" and illuminate all of the exterior of B except for four "quadrants" with apices at B. These quadrants can be illuminated with a $\frac{\pi}{2}$-floodlight placed at their apices, see Figure 1. Our result follows, as no point is illuminated by two $\frac{\pi}{2}$-floodlights. □

Theorems 1 and 2 imply the following theorem:

Theorem 3 *Let P be an orthogonal polygon with $|P| = n$. Then $\chi(P \cup ext(P), \frac{\pi}{2}) = 1$.*

Theorem 4 *Let P be an orthogonal polygon with $|P| = n$ and h holes. Then $2 \leq \chi(P, \frac{\pi}{2}) \leq h + 1$.*

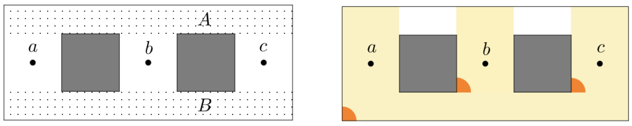
Proof. Consider the set of lines $\mathcal{L} = \{l_1, l_2, \ldots, l_k\}$ parallel to the x-axis that contain the *lowest bottom edges* of the holes of P, labelled in such a way that if $i < j$ the y-coordinate y_i of l_i is less than the y-coordinate y_j of l_j. Let l_0 be a lowest bottom edge of P and l_{k+1} a topmost edge of P. Then, for each $0 \leq i < k$, the set of points of P whose y coordinate belongs to the interval $[y_i, y_{i+1}]$ forms a set P_i of subpolygons of P. For each $i = 0, \ldots, k$ use Theorem 1 to illuminate all the subpolygons of P_i with color i, this can be done since all the elements in each P_i are pairwise independent. Since $k \leq h$, we use at most $h+1$ colors to illuminate P. For the lower bound consider Figure 2. Observe that when we illuminate the points a, b, and c either the region A or the region B, say A, will have two zones colored with color one and between them a third zone C not illuminated. In oder to illuminate C a second color must be used, since the visibility polygon of any floodlight that illuminates C overlaps at least one of the illuminated zones of A. □

Theorems 4 and 2 imply the following theorem:

Theorem 5 *Let P be an orthogonal polygon with $|P| = n$ and h holes. Then $2 \leq \chi(P \cup ext(P), \frac{\pi}{2}) \leq h + 1$.*

Theorem 6 *Let P be an orthogonal polygon with $|P| = n$. Then $\chi(P, \pi) = 2$.*

Proof. We place π-floodlights into P using the Theorem 1 algorithm with the following changes: In steps 1 to 3 we use color one and 0 orientation on the π-floodlights placed in the initial edge and the lower windows. In step 4 we use color two on the π-floodlights that we place in the polygons P'' of the recursive step, alternating between color one and color two each time we call the recursion. An intersection between visibility polygons is generated when we place a π-floodlight in a P'' polygon that has two



(a) (b)

Figure 2: (a) An orthogonal polygon P with holes (in gray) s.t. $2 \leq \chi(P, k\frac{\pi}{2})$, $k = 1, 2$. This family grows by adding holes to the polygon. (b) If points a, b, and c are illuminated with color one, then either the region A or the region B, has at least two illuminated zones, and between them, a not illuminated zone, which forces the use of a second color to illuminate the polygon.

edges that are P' windows, which is not a problem because they have different colors. For lack of space we omit the proof for the lower bound of our result. □

Theorem 7 *Let P be an orthogonal polygon with $|P| = n$ and h holes. Then $2 \leq \chi(P, \pi) \leq 2(h + 1)$.*

Proof. The proof is the same as that of Theorem 4 by substituting Theorem 1 for Theorem 6. For the lower bound we only use π-floodlights instead of $\frac{\pi}{2}$-floodlights. For the upper bound, the substitution of 1 for Theorem 6 works because the remaining polygons have no holes and can be illuminated using Theorem 6, which is used to illuminate orthogonal polygons without holes using π-floodlights. By Theorem 6 we need two colors, so the upper bound is $2(h + 1)$. □

5 Orthogonal illumination of orthogonal polyhedra with α-segments of size $\pi/2$ and π

Observe first that any orthogonal polyhedron P is the union of lifting polyhedra with pairwise disjoint interiors.

Let $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_k\}$ be the set of planes containing the faces of P parallel to the xy-plane, s.t. $i < j$ iff the z coordinate z_i of Q_i is less than the z coordinate z_j of Q_j. Then, for each $1 \leq i \leq k-1$, the set of points of P whose z coordinate belongs to the interval $[z_i, z_{i+1}]$ form a lifting orthogonal polyhedron P_i. Evidently $P = P_1 \cup \ldots \cup P_{k-1}$.

Let $\mathcal{Q}' = \{Q'_1, Q'_2, \ldots, Q'_{k-1}\}$ be a set of planes parallel to the xy-plane, such that Q'_i intersects P_i midway between Q_i and Q_{i+1}. Consider the plane $Q' \in \mathcal{Q}'$ such that the orthogonal polygon $Q' \cap P$ maximizes the number h_{xy} of holes it has. Define in similar way h_{xz} and h_{yz}, and let $h = \min\{h_{xy}, h_{xz}, h_{yz}\}$.

Theorem 8 *If $h = 0$ then $\chi(P, \frac{\pi}{2}) = 1$, and $\chi(P, \pi) \leq 2$. If $h > 0$ then $\chi(P, \frac{\pi}{2}) \leq h + 1$ and $\chi(P, \pi) \leq 2(h + 1)$.*

Proof. We will sketch the proof for $\chi(P, \frac{\pi}{2}) = 1$, and $h = 0$. The others are done in a similar way. Observe that each P_i as defined above is a lifting orthogonal polyhedron. We use $\frac{\pi}{2}$-segments to illuminate it as follows: Let P'_i be the orthogonal polygon obtained by intersecting Q'_i with P_i. Observe that any placement of $\frac{\pi}{2}$-floodlights that illuminates P'_i can be transformed into a set of $\frac{\pi}{2}$-segments that illuminate P_i, each of length $z_{i+1} - z_i$, and perpendicular to the xy-plane. By Theorem 1 one such set with $\chi(P, \frac{\pi}{2}) = 1$ exists. This induces a set of $\frac{\pi}{2}$-segments that illuminates P_i for which $\chi(P, \frac{\pi}{2}) = 1$. Our result follows. □

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